# ON SOME PROBLEMS OF KOROVKIN THEORY IN HARMONIC ANALYSIS

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**MANJU RANI AGRAWAL**

*to the*

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Dedicated to

my

Satgurudev.

# CERTIFICATE

It is to certify that the work contained in the thesis entitled "On some problems of Korovkin theory in Harmonic analysis" by Manju Rani Agrawal, has been carried out under my supervision and has not been submitted elsewhere for a degree.

U.B. Tewari

( U. B. Tewari )
Department of Mathematics
I.I.T. Kanpur

December, 1993

# ABSTRACT

This thesis is concerned with some problems of Korovkin approximation theory in certain commutative Banach algebras studied in Harmonic Analysis. Study of universal Korovkin sets in commutative Banach algebras with respect to various classes of operators was initiated by Altomare in 1982. An interesting open problem in Korovkin approximation theory has been that of characterization of commutative Banach algebras A with a continuous involution, in terms of their maximal ideal space $\Delta A$, which admit a finite universal Korovkin set w.r.t. positive operators. Answer is known only for some particular Banach algebras.

In Chapter III, we solve the problem for the following commutative Banach algebras : (i) The group Algebra $L^1(G)$, where G is a locally compact abelian group. (ii) The centre $Z(L^1(G))$ of the group algebra, where G is a compact group. (iii) $Z(L^1(G))$, where G is a connected [Z] group. (iv) $Z(L^1(G))$, where G is a [Z] group having a compact open normal subgroup K such that G = KZ.

In Chapter IV, we solve the problem for (i) a Segal algebra S(G) which is closed under involution, where G, a compact abelian group. (ii) Z(S(G)), the centre of a Segal algebra S(G) which is closed under involution, and G, a compact group.

If G is a non compact locally compact abelian group then the problem for the Segal algebras on G is not easy. In Chapter V, we solve this problem w.r.t. positive spectral contraction operators. We also solve the problem w.r.t. positive spectral contraction operators on Z(S(G)), where G is a [Z] group containing a compact open normal subgroup K such that G = KZ.

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( Manju R. Agrawal )

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# CHAPTER I

# INTRODUCTION

It was P.P. Korovkin who proved in 1953 ([30]) the following theorem which is as simple in its statement as it is surprising in its conclusion:

Suppose $\{T_n\}_{n=1}^{\infty}$ is a sequence of positive linear operators on $C([0,1],\mathbb{R})$, the space of all real valued continuous functions on the interval $[0,1]$, such that $\lim_{n\to\infty} T_n h = h$ uniformly on $[0,1]$ for every $h \in \left\{1,x,x^2\right\}$. Then $\lim_{n\to\infty} T_n f = f$ uniformly for every $f \in C([0,1],\mathbb{R})$.

Since the appearance of this result, a large number of mathematicians have worked in order to extend the Korovkin's theorem to other function spaces or more generally to abstract spaces. They adapted many directions such as Banach lattices ([11], [15], [18], [19], [58] etc.), $L^p$-spaces and Banach spaces ([14], [17], [29], [31], [36] etc.), non-commutative *-algebras and Banach algebras ([10], [33], [50], [53], [57] etc.) commutative *-algebras and Banach algebras ([1], [2], [41], [43], [47] etc.) and so on. Work done along these lines during the last 40 years has developed into a branch of abstract approximation theory which is known as Korovkin-type Approximation Theory. Quantitative versions estimating the rate of convergence (see for

example [13], [16], [38], [42] etc.) as well as qualitative versions trying to investigate the abstract background responsible for the validity of the theorem have appeared in the literature. We cite the paper of Altomare and Campiti [5] as a nice systematic collection of references for the qualitative Korovkin-type results. The paper by Pannenberg [44] is an excellent survey article on the subject.

In this thesis we shall be concerned with the Korovkin approximation theory in commutative Banach algebras initiated by Altomare in 1982 [1]. It was further explored and enriched by Altomare, Pannenberg, Romanelli, Beckhoff and many other authors.

The notion of universal Korovkin sets in commutative Banach algebras with respect to various classes of operators has been introduced in [1], [2], [3] and [47]. We shall restrict ourselves to universal Korovkin sets w.r.t. positive operators, w.r.t. positive contraction operators and w.r.t. positive spectral contraction operators. A subset S of a commutative Banach algebra A with a continuous involution is said to be a universal Korovkin set w.r.t. positive operators if B is a commutative Banach algebra with continuous symmetric involution, $T : A \longrightarrow B$ is an algebraic *-homomorphism and if a uniformly bounded net $\{T_\alpha\}$ of positive operators from A to B satisfies $\lim_\alpha \|(T_\alpha x - Tx)\hat{}\|_\infty = 0 \quad \forall\ x \in S$ then $\lim_\alpha \|(T_\alpha y - Ty)\hat{}\|_\infty = 0 \quad \forall\ y \in A$ ($\|\hat{x}\|_\infty$ denotes the spectral radius of x). Universal Korovkin set w.r.t. positive contraction operators and universal Korovkin set w.r.t. positive spectral

contraction operators are defined in a similar fashion. In terms of universal Korovkin sets, Korovkin's original theorem may be restated in its generalized form (see [1]) :

The set $\left\{1,\ x,\ x^2\right\}$ is a universal Korovkin set in the Banach algebra $C([0,1])$ w.r.t. positive operators.

In fact, the situation is interesting and useful if there exists a finite universal Korovkin set. An eminent problem in Korovkin theory has been (see [44], [46]) that of characterization of the commutative Banach algebras A, in terms of their maximal ideal space $\Delta A$, which admit a finite universal Korovkin set. (The problem stands as such with respect to each class of operators under consideration). The problem still remains unsolved, in general. However, answer is known for some particular Banach algebras. An excellent account of the information regarding the problem may be found in [46].

For a compact Hausdorff space X, the Banach algebra C(X) admits a finite universal Korovkin set w.r.t. positive operators iff X is metrizable and has finite covering dimension. This seems to have been observed for the first time by Saskin [56]. In fact, a commutative $C^*$-algebra A admits a finite universal Korovkin set w.r.t. positive operators iff $\Delta A$ is a finite dimensional separable metric space. In [45], Pannenberg has shown that the group algebra $L^1(G)$, where G is a locally compact abelian group, admits a finite universal Korovkin set w.r.t. positive contraction operators iff $\hat{G}$ (the dual group of G) is finite dimensional

separable metric space. On the other hand, there are Banach algebras which do not have any finite universal Korovkin sets. For example, if G is a non discrete locally compact abelian group then the convolution algebra of measures, M(G), can never possess a finite universal Korovkin set w.r.t. spectral contraction operators. See [46] for many other such examples.

Our main objective in this thesis will be to investigate the problem (mentioned above), where the Banach algebra A is varying over some well known commutative Banach algebras studied in Harmonic Analysis. In Chapter II, we explain the notations and give a brief account of the definitions and preliminary results from Harmonic Analysis and Korovkin theory which we shall need in the sequel.

In Chapter III, we define a notion of property 'P' in commutative Banach algebras with a continuous involution. Property 'P' is satisfied, in particular, by commutative Banach algebras with continuous symmetric involution and a bounded approximate identity. For a commutative Banach algebra A having a continuous symmetric involution and satisfying property 'P', we prove that A admits a finite universal Korovkin set w.r.t. positive operators iff there exist finitely many elements in A such that their Gelfand transforms separate points of $\Delta A$. It turns out that if A is such a Banach algebra then A has a finite universal Korovkin set w.r.t. positive operators iff it has a finite universal Korovkin set w.r.t. positive contraction operators. Thus, in view of results proved by Pannenberg in [45],

we obtain a characterization of locally compact abelian groups G for which $L^1(G)$ possesses a finite universal Korovkin set w.r.t. positive operators.

It is known ([39]) that for a locally compact group G the centre $Z(L^1(G))$ of $L^1(G)$ is nontrivial iff G has a compact invariant neighborhood of identity. If G is such a group then $Z(L^1(G))$ has a continuous symmetric involution and bounded approximate identity ([35]). In Section 2 of Chapter III, we prove that for a compact group G, $Z(L^1(G))$ has a finite (three element) universal Korovkin set w.r.t. positive operators iff G is metrizable. As a consequence of this result we obtain that for a connected [Z] group G, $Z(L^1(G))$ admits a finite universal Korovkin set w.r.t. positive operators iff G is metrizable.

In the last part of Chapter III, we are concerned with the case of non-connected central topological groups. We prove that if G is a [Z] group having a compact open normal subgroup K such that G = KZ then following are equivalent :

(1) $Z(L^1(G))$ has a finite universal Korovkin set w.r.t. positive operators.

(2) G is separable metrizable and G/K has finite torsion free rank.

(3) $\hat{G}$ is finite dimensional separable metric space.

We also prove that if G is a separable metrizable periodic [Z] group then $Z(L^1(G))$ admits a finite (four element) universal Korovkin set w.r.t. positive operators.

Let $S(G)$ denote a Segal algebra over a locally compact group $G$ such that it is closed under involution (inherited from $L^1(G)$). If $G$ is abelian then $S(G)$ is a commutative Banach algebra with continuous symmetric involution ([51]). If $G$ is non abelian then $Z(S(G))$ is nontrivial iff $G$ has a compact invariant neighborhood of identity ([48]). If $G$ is such a group then $Z(S(G))$ is a commutative Banach algebra with continuous symmetric involution. In Chapter IV, we investigate the problem in these commutative Banach algebras. We show that a Segal algebra need not satisfy property 'P' and therefore the techniques developed in Chapter III are no longer at our disposal. However, we prove by different techniques that if $G$ is a compact abelian group then $S(G)$ admits a finite (three element) universal Korovkin set w.r.t. positive operators iff $G$ is metrizable. We also prove by similar techniques that for a compact group $G$, $Z(S(G))$ has a finite (three element) universal Korovkin set w.r.t. positive operators iff $G$ is metrizable.

In [2], Altomare has proved that if $A$ is a commutative Banach algebra with continuous symmetric involution and a bounded approximate identity uniformly bounded by one and if $S$ is a subset of $A$ such that $\hat{S}$ separates points of $\Delta A$ strongly then $S \cup SS^*$ is a universal Korovkin set w.r.t. positive contraction operators. We extend this result to the commutative Banach algebras with continuous symmetric involution and satisfying a certain property '$P_1$' (to be specified in Chapter IV). We also show by an example that in the absence of property '$P_1$' the result need not be true.

Fifth chapter of the thesis is devoted to the study of universal Korovkin sets w.r.t. positive spectral contraction operators in a commutative Banach algebra A with continuous symmetric involution. We observe that if A satisfies property '$P_1$' then universal Korovkin sets w.r.t. positive spectral contraction operators coincide with universal Korovkin sets w.r.t. positive contraction operators. In the absence of property '$P_1$' we show by an example that the two notions are different. In [55, Theorem 1.1(2)] Romanelli has proved that if A is a commutative Banach algebra with continuous symmetric involution and bounded approximate identity then a subset S of A is a universal Korovkin set w.r.t. positive spectral contraction operators iff $\hat{S}$ is a universal Korovkin set in $C_o(\Delta A)$ w.r.t. positive contraction operators. By techniques similar to those used by Romanelli in Theorem 1.1(2) of [55], we prove that the conclusion of Romanelli remains valid if the condition that A has a bounded approximate identity is omitted. This enables us to obtain a universal Korovkin set w.r.t. positive spectral contraction operators in a Segal algebra S(G) over a locally compact abelian group via the techniques available for constructing universal Korovkin sets w.r.t. positive contraction operators in $C_o(\hat{G})$. We prove that S(G) admits a finite universal Korovkin set w.r.t. positive spectral contraction operators iff $\hat{G}$ is finite dimensional separable metric space.

# CHAPTER II

# PRELIMINARIES

In this chapter we explain the notations used and give a brief account of the definitions and preliminary results to be used in the sequel. For standard results from Harmonic Analysis the reader is referred to the books by Hewitt and Ross [26] and Reiter [51], [52].

## 2.1 Generalities

Throughout the thesis G will denote a locally compact Hausdorff group. $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{T}$ will denote respectively the group of real numbers, the group of integers and the circle group. $\mathbb{N}$ and $\mathbb{C}$ respectively denote the set of natural numbers and the set of complex numbers. For a Banach space B, $B'$ will denote its dual. For a locally compact Hausdorff space X, $C_o(X,\mathbb{C})$ $(C_o(X,\mathbb{R}))$ will denote the Banach space of all complex valued (real valued) continuous functions on X vanishing at infinity. We shall usually write $C_o(X)$ for $C_o(X,\mathbb{C})$. If X is compact then we write simply $C(X)$ for $C_o(X)$. We say that a subset S of $C_o(X)$ separates points of X if for $x_1, x_2 \in X$, $h(x_1) = h(x_2)$ $\forall\, h \in S$ implies $x_1 = x_2$; S is said to separate points of X strongly if in addition, $\forall\, x \in X$, there exists an $h \in S$ such that $h(x) \neq 0$.

$\mathcal{M}_b(X)$ will denote the set of all bounded regular Borel measures on X. Set of all positive bounded regular Borel measures on X will be denoted by $\mathcal{M}_b^+(X)$. For $x \in X$, $\delta_x$ will denote the point measure of mass 1, concentrated at x. For $\mu \in \mathcal{M}_b(X)$ the support of $\mu$ is denoted by $\mathrm{Supp}(\mu)$.

If B is a Banach space and $\phi \in B'$ then the symbols $\phi(f)$ and $\langle f,\phi\rangle$ both will indicate the value of $\phi$ at f.

If h is a function defined on Y and $X \subset Y$ then $h|_X$ will denote the restriction of h to X. As usual $\delta_{mn}$ will stand for the Kronecker delta function.

By dimension of a topological space X we shall mean the covering dimension. This is defined as follows (see [40], p. 9).

Let $\mathcal{U}$ be a covering of a topological space X. We say that $\mathcal{U}$ has order n (and write ord $\mathcal{U} = n$) if for each $x \in X$, number of members in $\mathcal{U}$ containing x is at most n.

We say that $\dim X \le n$ if every finite open covering $\mathcal{U}$ of X has an open refinement $\mathcal{B}$ such that ord $\mathcal{B} \le n+1$. If it is true that $\dim X \le n$ and it is false that $\dim X \le n-1$ then we say $\dim X = n$. If $\dim X \le n$ is false for each integer n then $\dim X = \infty$.

## 2.2 Commutative Banach algebras

All the Banach algebras considered will be complex Banach algebras. For standard results in the theory of commutative Banach algebras we refer the book by Larsen [32]. For any algebra A, Z(A) will denote its centre. For a commutative Banach algebra

$A$, $\Delta A$ will denote its maximal ideal space, the set of all nontrivial complex homomorphisms of $A$. If $A$ has an identity then $\Delta A$ is a $\omega^*$-compact subset of $A'$. In general $\Delta A$ is locally compact. If $\Delta A$ is not compact, we denote by $\Delta_\omega A$ the one point compactification of $\Delta A$. For $x \in A$, $\hat{x}$ and $\|\hat{x}\|_\infty$ will denote respectively the Gelfand transform and the spectral radius of $x$. It is well known that $\hat{A} = \{\hat{x} : x \in A\}$ is a point separating subalgebra of $C_o(\Delta A)$. $A$ is said to be semisimple if the map $x \longrightarrow \hat{x}$ is one to one.

$A$ is said to have an involution if there exists a map $* : A \longrightarrow A$ such that

(i) $(x+y)^* = x^*+y^*$

(ii) $(xy)^* = y^*x^*$

(iii) $x^{**} = x$

and (iv) $(\alpha x)^* = \bar{\alpha}x^*$ for every $x,y \in A$ and $\alpha \in \mathbb{C}$.

If $A$ has an involution then elements of the form $x^*x$ are called positive elements. Involution on $A$ is said to be symmetric if $\hat{x^*} = \bar{\hat{x}}$ holds $\forall\ x \in A$. If $A$ has a symmetric involution then it can be easily seen that $\hat{A}$ is dense in $C_o(\Delta A)$. $A$ is called a $C^*$-algebra if $\|xx^*\| = \|x\|^2$ holds $\forall\ x \in A$.

Let $A$ and $B$ be commutative Banach algebras. A linear operator $T : A \longrightarrow B$ is said to be spectral contraction (contraction) if $\|(Tx)^\wedge\|_\infty \le \|\hat{x}\|_\infty$ $(\|Tx\| \le \|x\|)$ holds $\forall\ x \in A$.

Suppose that A and B are commutative Banach algebras with continuous involution. A linear operator $T : A \longrightarrow B$ is said to be positive if for every $x \in A$, $T(x^*x)$ is a positive element of B. In particular, a linear functional $\phi$ on A is called positive if $\phi(x^*x) \geq 0 \ \forall \ x \in A$. All the operators and functionals considered in the thesis will be linear. For the sake of brevity we shall simply write them as operator and functional.

If A does not have an identity then a continuous positive functional $\phi$ on A is said to be extendable if it can be extended as a positive functional on $A_1$, where $A_1$ is the Banach algebra obtained from A by adjoining an identity.

A is said to have a bounded approximate identity if there is a uniformly bounded net $\{e_\alpha\}$ in A such that $\lim_\alpha \|e_\alpha x - x\| = 0 \ \forall \ x \in A$.

For a subset S of A, $SS^*$ will denote the set $\{xx^* : x \in S\}$ and if $x \in A$ then $xS$ will denote the set $\{xy : y \in S\}$.

## 2.3 Some special classes of groups

Let G be a locally compact Hausdorff group. The set of equivalence classes of continuous irreducible unitary representations of G is denoted by $\hat{G}$ and is called the dual object of G. If G is abelian then $\hat{G}$ is identified with the dual group of G ([26], §27.4(a)). If G is compact then every continuous irreducible unitary representation of G is finite dimensional. For $\sigma \in \hat{G}$, $d_\sigma$ denotes the (finite) dimension of the representation space $H_\sigma$. $I_\sigma$ denotes the identity operator on $H_\sigma$ and $\chi_\sigma$ denotes

the character associated with $\sigma$. $u_{ij}^{(\sigma)}$ $(i,j \in \{1,\ldots,d_\sigma\})$ will denote the coordinate functions for $\sigma$ and an arbitrary but fixed orthonormal basis in $H_\sigma$. The linear space spanned by the coordinate functions is denoted by $\mathcal{T}(G)$ and is called the space of trigonometric polynomials. For the details we refer to Section 27 and Section 28 of Hewitt and Ross [26].

We shall be needing several classes of locally compact groups. For the following definitions see [24].

(i) $G \in [IN]$ iff G has a compact invariant neighborhood of the identity.

(ii) $G \in [SIN]$ iff G has a fundamental system of invariant neighborhoods of the identity, that is, every neighborhood of the identity contains an invariant neighborhood of the identity.

(iii) $G \in [FIA]^-$ iff $G \in [SIN]$ and the sets $\left\{xyx^{-1} : x \in G\right\}^-$ are compact for each $y \in G$.

(iv) $G \in [Z]$ (or G is a central topological group) iff $G/Z(G)$ is compact, where $Z(G)$ is the centre of G.

These classes satisfy the following inclusion relation (see, [24]).

$$[Z] \subsetneq [FIA]^- \subsetneq [SIN] \subsetneq [IN].$$

It is known ([22]) that if G is a [Z] group then every continuous irreducible unitary representation of G is finite dimensional. So we use the same notations as for the compact case. If $\sigma \in \hat{G}$ and K is a subgroup of G then $\sigma|_K$ will denote the

restriction of $\sigma$ on K.

$x \in G$ is said to be a periodic element if $[x]$, the closed subgroup generated by x, is compact. G is called periodic if it consists of periodic elements.

The commutator subgroup of G is the subgroup generated by the set $\left\{xyx^{-1}y^{-1} : x,y \in G\right\}$ and is denoted by $G'$. If $K_1,\ldots,K_n$ are subsets of G, $\langle K_1,\ldots,K_n\rangle$ will denote the subgroup generated by these subsets.

2.4

On every locally compact group G, there exists a unique (up to scalar multiples) left translation invariant measure $\lambda$, called the left Haar measure. If G is compact or discrete, Haar measure will be normalized in the standard way. That is, for G compact, Haar measure is normalized in such a way that $\lambda(G) = 1$ and if G is discrete then it is normalized so that $\lambda(\{x\}) = 1 \ \forall \ x \in G$. Usually we shall write dx for $d\,\lambda(x)$. For $1 \le p < \infty$, $L^p(G)$ will denote the Banach space of all functions f on G such that $|f|^p$ is integrable with respect to left Haar measure. $L^\infty(G)$ will stand for the Banach space of all essentially bounded functions on G.

If $f, g \in L^1(G)$ then the convolution of f and g is defined by $f * g(y) = \int_G f(x)\, g(x^{-1}y)dx$. If G is abelian then $L^1(G)$ is a commutative semisimple Banach algebra with convolution as multiplication. For $f \in L^1(G)$, define $f^\sim$ by $f^\sim(x) = f(x^{-1})$. Then $f \longrightarrow f^\sim$ is a continuous symmetric involution on $L^1(G)$. Furthermore, $L^1(G)$ admits a bounded approximate identity and

maximal ideal space of $L^1(G)$ can be identified with $\hat{G}$, the dual group of G, see [32].

If G is non abelian then $Z(L^1(G))$ is nontrivial iff $G \in [IN]$, see [39]. If G is such a group then $Z(L^1(G))$ is a commutative semisimple Banach algebra with convolution as multiplication. Moreover, it has a continuous symmetric involution ~ (defined as above) and a bounded approximate identity (see, [35]).

M(G) will denote the Banach algebra of all bounded regular Borel measures on G. For $\mu \in M(G)$, $\hat{\mu}$ will denote its Fourier-Stieltjes transform.

Note that we are using the symbol * to denote convolution and involution both. However, it will be clear from the context whether it is being used for convolution or involution.

## 2.5 Segal algebras

Let G be a non discrete locally compact group. A dense subalgebra S(G) of $L^1(G)$ is called a Segal algebra if

(i) S(G) is left translation invariant, that is, for every $x \in G$ and $f \in S(G)$ ${}_xf$ also belongs to S(G). (${}_xf$, the left translate of f, is defined by ${}_xf(y) = f(x^{-1}y)$).

(ii) S(G) itself is a Banach algebra with a norm $\| \ \|_S$. The norm $\| \ \|_S$ is left invariant, that is, $\|{}_xf\|_S = \|f\|_S \ \forall f \in S(G)$ and $x \in G$.

(iii) For each $f \in S(G)$, the map $x \longrightarrow {}_xf$ from G into S(G) is continuous.

If G is abelian then S(G) is a commutative semisimple Banach algebra and its maximal ideal space can be identified with $\hat{G}$. S(G) has a continuous symmetric involution provided it is closed under the involution (inherited from $L^1(G)$). A proper Segal algebra can never admit a bounded approximate identity. The basic references for Segal algebras are the books by Reiter ([51],[52]). We now give some examples of Segal algebras.

(1) $L^1(G) \cap C_o(G)$ and $L^1(G) \cap L^p(G)$ with norms $\|f\|_S = \|f\|_1 + \|f\|_\infty$ and $\|f\|_S = \|f\|_1 + \|f\|_p$ $(1 < p < \infty)$.

(2) G compact; C(G) and $L^p(G)$ $(1 < p < \infty)$ with norms $\|\ \|_\infty$ and $\|\ \|_p$.

(3) If G is abelian and $\mu$ is a positive unbounded measure on $\hat{G}$ then $A^p_\mu(G) = \left\{f \in L^1(G) : \hat{f} \in L^p(\hat{G},\mu)\right\}$ $(1 \le p < \infty)$ with norm $\|f\|_1 + \|\hat{f}\|_{L^p(\hat{G},\mu)}$ is a Segal algebra. We simply write it $A_p(G)$ when $\mu$ is the Haar measure on $\hat{G}$.

## 2.6 Korovkin Theory in commutative Banach algebras

Since the appearance of the classical theorem of Korovkin in 1953 [30], many mathematicians have worked in order to extend the theorem to other function spaces or more generally to abstract spaces. They adapted many directions such as Banach lattices, Banach spaces, Banach algebras and so on.

In this thesis we shall be concerned with the Korovkin theory in commutative Banach algebras. Study of universal Korovkin sets in commutative Banach algebras was initiated by Altomare (see, [1],[2], [3]). It has been nurtured and enriched by Altomare,

Pannenberg, Romanelli, Beckhoff and many other authors.

Let A be a commutative Banach algebra with a continuous involution and S be a subset of A. $Kor_u^+(S)$, the universal Korovkin closure of S with respect to positive operators is defined as follows (see, [2] and [3]):

$Kor_u^+(S) = \{y \in A$ : for every commutative Banach algebra B with continuous symmetric involution, every algebraic *-homomorphism $T : A \longrightarrow B$ and every uniformly bounded net $\{T_\alpha\}$ of positive operators from A to B, $\lim_\alpha \|(T_\alpha x - Tx)\hat{}\|_\infty = 0$ for all $x \in S$ implies $\lim_\alpha \|(T_\alpha y - Ty)\hat{}\|_\infty = 0\}$.

If $Kor_u^+(S) = A$, we say that S is a universal Korovkin set in A w.r.t. positive operators, (in case when S is a linear subspace of A, S will be referred as universal Korovkin space w.r.t. positive operators).

Universal Korovkin closure and universal Korovkin sets w.r.t. positive contraction operators and w.r.t. positive spectral contraction operators are defined similarly (one must replace 'positive operators' by the respective class of operators in the definition). For the definition of contraction and spectral contraction operators, see §2.2. $Kor_u^{1+}(S)$ and $\hat{K}or_u^{1+}(S)$ respectively will denote universal Korovkin closure of S w.r.t. positive contraction operators and universal Korovkin closure of S w.r.t. positive spectral contraction operators. If A has an identity then $Kor_u^+(S)$, $Kor_u^{1+}(S)$ and $\hat{K}or_u^{1+}(S)$ are defined by requiring that B has an identity and T is unital. If A is a

commutative $C^*$-algebra with an identity then the three notions $\mathrm{Kor}_u^+(S)$, $\mathrm{Kor}_u^{1+}(S)$ and $\hat{\mathrm{K}}\mathrm{or}_u^{1+}(S)$ coincide (cf. [3], Theorem 1).

Let E be a Banach space such that a partial order is defined on it. A subset S of E is said to be a Korovkin set in E w.r.t. positive operators if for every uniformly bounded net $\{T_\alpha\}$ of positive operators on E, $\lim_\alpha \|T_\alpha h - h\| = 0 \ \forall \ h \in S$ implies $\lim_\alpha \|T_\alpha f - f\| = 0 \ \forall \ f \in E$.

If $E = C_o(X)$ (X locally compact Hausdorff space) then a subset S of $C_o(X)$ is a Korovkin set w.r.t. positive operators iff S is a universal Korovkin set in $C_o(X)$ w.r.t. positive operators (see [3], Theorem 1).

In terms of so defined notion of Korovkin sets, Korovkin's original theorem may be restated as:

The set $\left\{1,x,x^2\right\}$ is a Korovkin set in $C([0,1],\mathbb{R})$ w.r.t. positive operators.

Korovkin's theorem has been generalized to the Banach space $C(X,\mathbb{R})$, where X is a compact Hausdorff space, by many authors. The following result has appeared in the work of Bauer [6], [7], Berens and Lorentz [12], Saskin [56] and others.

Theorem 2.6.1 Let X be a compact Hausdorff space and S be a subset of $C(X,\mathbb{R})$ containing the constant function 1. Then S is a Korovkin set in $C(X,\mathbb{R})$ w.r.t. positive operators $\Longleftrightarrow$ $X = \Big\{x \in X :$ if $\mu \in \mathcal{M}_b^+(X)$ satisfies $\mu(h) = h(x) \ \forall \ h \in S$ then $\mu = \delta_x\Big\}$.

The set on the right in Theorem 2.6.1 is referred as the Choquet boundary of S (or Choquet boundary of H, where H is the linear hull of S, and is denoted by Ch(H), see [49]).

Theorem 2.6.1 was extended to the complex valued case by Altomare and Boccaccio in [4].

Theorem 2.6.2 (Altomare and Boccaccio [4], Theorem 2.1) Let X be a compact Hausdorff space and S be a subset of $C(X,\mathbb{C})$ containing the constant function 1. Then S is a Korovkin set in $C(X,\mathbb{C})$ w.r.t. positive operators iff

$$X = \left\{x \in X: \text{ if } \mu \in \mathcal{M}_b^+(X) \text{ satisfies } \mu(h) = h(x)\ \forall\ h \in S \text{ then } \mu=\delta_x\right\}.$$

Theorem 2.6.1 was extended to $C_o(X,\mathbb{R})$, where X is a locally compact Hausdorff space, by Bauer and Donner in [8].

Theorem 2.6.3 (Bauer and Donner [8], Corollary 4) Let X be a locally compact Hausdorff space and S be a subset of $C_o(X,\mathbb{R})$. Then S is a Korovkin set in $C_o(X,\mathbb{R})$ w.r.t. positive operators $\Longleftrightarrow$

$$X = \left\{x \in X: \text{ if } \mu \in \mathcal{M}_b^+(X) \text{ satisfies } \mu(h) = h(x)\ \forall\ h \in S \text{ then } \mu=\delta_x\right\}.$$

Among many interesting applications following result is also proved in [8].

Theorem 2.6.4 (Bauer and Donner [8], Prop. 4.3) Let X be a locally compact Hausdorff space and $X_\omega$ be its one point compactification. Let H be a subspace of $C(X_\omega,\mathbb{R})$, $f \in C_o(X,\mathbb{R})$ be strictly positive and $H_f$ be defined as $H_f = \left\{h(f|_X) : h \in H\right\}$. If $X \subset Ch(H)$, the Choquet boundary of H, then $H_f$ is a Korovkin space in $C_o(X,\mathbb{R})$.

Complex valued analogue of Theorem 2.6.3 is a particular case of the following general result due to Altomare [2].

Theorem 2.6.5 (Altomare [2], Cor. 1.2) Let A be a commutative Banach algebra with a continuous symmetric involution and S be a subset of A. Then S is a universal Korovkin set in A w.r.t. positive operators iff for every $m \in \Delta A$ and for every continuous positive functional $\phi$ on A, the relation $\phi(x) = m(x)\ \forall\ x \in S$ implies $\phi(y) = m(y)\ \forall\ y \in A$.

Korovkin sets w.r.t. positive contraction operators in the space $C_o(X,\mathbb{R})$ have been studied in Bauer and Donner [9] and Bauer [6]. We now list some results regarding universal Korovkin closure and universal Korovkin sets w.r.t. positive contraction operators which we shall be needing in the sequel.

Theorem 2.6.6 (Altomare [2], Theorem 4.1(1)) Let A be a commutative Banach algebra with continuous symmetric involution and S be a subset of A. Then

$$\mathrm{Kor}_u^{1+}(S) = \Big\{ y \in A : \text{if } m \in \Delta_\omega A \text{ and } \phi \text{ is a continuous positive functional on } A \text{ such that } \|\phi\| \le 1 \text{ and if } \phi(x) = m(x) \text{ for all } x \in S \text{ then } \phi(y) = m(y) \Big\}.$$

Following theorem is a particular case of Theorem 2.6.6 for $A = C_o(X)$.

Theorem 2.6.7 (Altomare [2], Theorem 4.2) Let X be a locally compact Hausdorff space and $X_\omega$ be its one point compactification. Let S be a subset of $C_o(X)$. Then

$$\mathrm{Kor}_u^{1+}(S) = \Big\{ f \in C_o(X) : \text{if } \mu \in \mathcal{M}_b^+(X) \text{ with } \|\mu\| \le 1 \text{ and } x \in X_\omega \text{ and if } \mu(h) = h(x)\ \forall\, h \in S, \text{ then } \mu(f) = f(x) \Big\}.$$

**Theorem 2.6.8** (Altomare [2], Corollary 4.3) Let $S$ be a subset of $C_o(X)$ such that it separates points of $X$ strongly. Then $S \cup |S|^2$ is a universal Korovkin set in $C_o(X)$ w.r.t. positive contraction operators. $\left(|S|^2 = \{|f|^2 : f \in S\}\right)$.

Following result will be useful for constructing universal Korovkin sets w.r.t. positive contraction operators.

**Theorem 2.6.9** (Altomare [2], Corollary 4.5) Let $A$ be a commutative Banach algebra with continuous symmetric involution and a bounded approximate identity $\{e_\alpha\}$ such that $\sup\limits_\alpha \|e_\alpha\| \le 1$. If $S$ is a subset of $A$ such that $\hat{S}$ separates points of $\Delta A$ strongly. Then $S \cup SS^*$ is a universal Korovkin set in $A$ w.r.t. positive contraction operators.

Furthermore, if $S = \{x_1, \ldots, x_n\}$ is finite, then $\left\{x_1, \ldots, x_n, \sum\limits_{i=1}^{n} x_i x_i^*\right\}$ is a universal Korovkin set in $A$ w.r.t. positive contraction operators.

# CHAPTER III

## KOROVKIN THEORY FOR POSITIVE OPERATORS ON GROUP ALGEBRAS

Universal Korovkin sets w.r.t. positive contraction operators have been extensively studied, see, for example, Altomare [2], Pannenberg [45], [46], Pannenberg and Romanelli [47]. However, not much information about universal Korovkin sets w.r.t. positive operators is available in the literature. In this chapter we attempt to fill this gap and provide further information which, in particular, applies to the Korovkin theory w.r.t. positive contraction operators as well. At this point it is worth mentioning that a universal Korovkin set w.r.t. positive contraction operators may fail to be universal Korovkin set w.r.t. positive operators. For example, the set of functions $\left\{x, x^2\right\}$ in $C_o((0,1])$ is not a universal Korovkin set w.r.t. positive operators though it is a universal Korovkin set w.r.t. positive contraction operators. In fact, a subset $S$ of $C_o((0,1])$ is a universal Korovkin set w.r.t. positive operators iff $\forall x_o \in (0,1]$ and for every positive bounded regular Borel measure $\mu$ on $(0,1]$, the equations $\mu(h) = h(x_o)$ $\forall$ $h \in S$, imply that $\mu = \delta_{x_o}$ (Theorem 2.6.5). If we take $x_o = 2/3$ and define $\mu$ as $\mu(f) = \frac{4}{3}\int_0^1 f(t)\, dt$ $\forall$ $f \in C_o((0,1])$, where dt denotes the Lebesgue measure restricted to

(0,1], then it is easy to check that $\mu(h) = h(2/3)\ \forall\ h \in \left\{x,x^2\right\}$ but $\mu \neq \delta_{2/3}$.

On the other hand, in view of Theorem 2.6.8, $\left\{x,x^2\right\}$ is a universal Korovkin set w.r.t. positive contraction operators since the function x separates points of (0,1] strongly.

Throughout this chapter A denotes a commutative Banach algebra with continuous and symmetric involution. The phrase 'universal Korovkin set' would mean 'universal Korovkin set w.r.t. positive operators' unless otherwise stated.

We shall say that a Banach algebra A with a continuous involution satisfies property 'P' if:

(P) Every continuous positive functional $\phi$ on A satisfies the inequality

$|\phi(x)| \leq C\|\hat{x}\|_\infty\ \forall\ x \in A$, for some positive constant C.

Note that a commutative Banach algebra with continuous symmetric involution and bounded approximate identity satisfies property 'P', (cf. [37], p. 275).

In fact, a commutative Banach algebra with continuous symmetric involution and such that its every continuous positive functional is extendable satisfies property 'P' ;

for if $\phi$ is a continuous positive functional on A which is extendable then there exists a constant C such that

$|\phi(x)|^2 \leq C\ \phi(x^*x)\ \forall\ x \in A$. [See, [26], Theorem 21.18,p.317].

Now proceeding as in Gelfand-Raikov-Shilov ([20], §8.14, p. 62), we get $\forall\ x \in A$

$$\begin{aligned} |\phi(x)| &\le C\|(x^*x)^\wedge\|_\infty^{1/2} \\ &= C\|\hat{x}\|_\infty, \text{ by symmetry of involution.} \end{aligned}$$

Motivated by the results proved by Bauer and Donner [8] for the algebra $C_o(X,\mathbb{R})$ we describe, in Section 1, a method for constructing universal Korovkin sets in a Banach algebra A with property 'P'. We use this to characterize the Banach algebras A with property 'P' which admit a finite universal Korovkin set w.r.t. positive operators. This characterization has many useful consequences. In particular, using results proved in [45], we obtain a characterization of the locally compact abelian groups G for which $L^1(G)$ admits a finite universal Korovkin set w.r.t. positive operators.

In Section 2, we characterize the compact groups G for which the commutative Banach algebra $Z(L^1(G))$ admits a finite universal Korovkin set. As a consequence we obtain a characterization of all the connected [Z] groups G for which $Z(L^1(G))$ admits a finite universal Korovkin set.

In Section 3, we deal with the situation when a [Z] group G is not connected and provide sufficient conditions in order that $Z(L^1(G))$ admits a finite universal Korovkin set. These conditions turn out to be necessary as well, for a large class of central topological groups. In fact, we prove the following theorem:

Theorem Let $G$ be a [Z] group having a compact open normal subgroup $K$ such that $G = KZ$. Then following are equivalent.

(1) $Z(L^1(G))$ admits a finite universal Korovkin set.

(2) $G$ is separable metrizable and $G/K$ has finite torsion free rank.

(3) $\hat{G}$ is finite dimensional separable metric space.

We also show that if $G$ is a separable metrizable periodic [Z] group then $Z(L^1(G))$ admits a finite universal Korovkin set.

§1

We shall prove the following Theorem,

Theorem 3.1.1 Let $A$ be a commutative Banach algebra with continuous symmetric involution and satisfying property 'P'. Then following are equivalent.

(i) $A$ admits a finite universal Korovkin set w.r.t. positive operators.

(ii) There exist finitely many elements in $A$ such that their Gelfand transforms separate points of $\Delta A$.

The proof depends on some intermediary results which we develope first. We shall need the following theorem due to Altomare [2] and some of its immediate consequences.

Theorem 3.1.2 [Altomare, [2], Corollary 1.2] Let $A$ be a commutative Banach algebra with continuous symmetric involution and $S$ be a subset of $A$. Let $\mathrm{Kor}_u^+(S)$ denote the universal Korovkin closure of $S$ in $A$ w.r.t. positive operators. Then

$$\mathrm{Kor}_u^+(S) = \Big\{y \in A : \text{if } \phi \text{ is a continuous positive functional on } A \text{ and } m \in \Delta A \text{ and if } \phi(x) = m(x) \text{ for all } x \in S, \text{ then } \phi(y) = m(y)\Big\}$$

**Corollary 3.1.1** Let X be a locally compact Hausdorff space and S be a subset of $C_o(X,\mathbb{C})$. Then

$$\mathrm{Kor}_u^+(S) = \Big\{f \in C_o(X,\mathbb{C}) : \text{if } x \in X \text{ and } \mu \in \mathcal{M}_b^+(X) \text{ satisfy } \mu(h) = h(x) \text{ for all } h \in S, \text{ then } \mu(f) = f(x)\Big\},$$

where $\mathcal{M}_b^+(X)$ denotes the set of all positive bounded regular Borel measures on X.

**Corollary 3.1.2** Let A be a commutative Banach algebra with continuous symmetric involution and satisfying property 'P'. Let S be a subset of A. Then S is a universal Korovkin set in A w.r.t. positive operators <==> $\hat{S}$ is a universal Korovkin set in $C_o(\Delta A)$ w.r.t. positive operators.

**Proof** Since A has symmetric involution, $\hat{A}$ is dense in $C_o(\Delta A)$, therefore, the implication '==>' is obviously true.

We now prove the reverse implication. Let $m \in \Delta A$ and $\phi$ be a continuous positive functional on A such that $\phi(x) = m(x)$ $\forall$ $x \in S$ holds. Since A satisfies property 'P', there exists a constant C such that

$$|\phi(x)| \leq C \, \|\hat{x}\|_\infty \ \forall \ x \in A.$$

Thus $\phi$ gives rise to a positive functional on $\hat{A}$ which is continuous w.r.t. supremum norm. Thus we get a measure $\mu \in \mathcal{M}_b^+(\Delta A)$

such that

$$\mu(\hat{x}) = \phi(x) \quad \forall\ x \in A.$$

In particular, we have $\mu(\hat{x}) = \hat{x}(m)\ \forall\ \hat{x} \in \hat{S}$. Since $\hat{S}$ is a universal Korovkin set in $C_o(\Delta A)$, it follows that $\mu = \delta_m$.

Therefore $\mu(\hat{y}) = \hat{y}(m) \quad \forall\ y \in A$

or $\qquad \phi(y) = m(y) \quad \forall\ y \in A.$

This completes the argument.

Let X be a locally compact Hausdorff space and $X_\omega$ be its one point compactification. Let f be a strictly positive function in $C_o(X,\mathbb{R})$. Let H be a subspace of $C(X_\omega,\mathbb{C})$ and $H_f$ denote the subspace of $C_o(X,\mathbb{C})$ defined as $H_f = \left\{f(h|_X) : h \in H\right\}$.

In view of Corollary 3.1.1, it can be easily seen that the following complex valued analogue of Lemma 4.2, Bauer and Donner [8], holds.

Proposition 3.1.1 $\quad Kor_u^+(H)$ and $Kor_u^+(H_f)$ satisfy the following inclusion,

$$(Kor_u^+(H))|_X \subseteq \frac{1}{f}\, Kor_u^+(H_f).$$

As a consequence we have the following (cf. Proposition 4.3, [8]).

Corollary 3.1.3 $\quad$ If $X \subseteq Ch(H)$, the Choquet boundary of H, then $H_f$ is a universal Korovkin space in $C_o(X,\mathbb{C})$ w.r.t. positive operators.

The proof of the following proposition is based on the techniques similar to those used by Altomare in [2, Corollary 4.3].

Proposition 3.1.2 Let S be a subset of $C_o(X,\mathbb{C})$ such that it separates points of X. Let $f \in C_o(X,\mathbb{R})$ be such that $f > 0$. Then $\{f\} \cup f\,S \cup |S|^2$ is a universal Korovkin set in $C_o(X,\mathbb{C})$ w.r.t. positive operators.

Proof Let S′ be the subset of $C(X_\omega,\mathbb{C})$ obtained by extending functions in S continuously to $X_\omega$. Let H denotes the linear hull of the set $\{1\} \cup f\,S' \cup f\,|S'|^2$ then $H_f$ is the subspace spanned by the set $\{f\} \cup f\,S \cup f\,|S|^2$. We show that $H_f$ is a universal Korovkin space in $C_o(X,\mathbb{C})$ w.r.t. positive operators. In view of Corollary 3.1.3, it suffices to show that $X \subseteq Ch(H)$. Let $x \in X$ and $\mu \in \mathcal{M}_b^+(X_\omega)$ such that $\mu(h) = h(x)\ \forall\ h \in H$. We must show that $\mu = \delta_x$.

Since $1 \in H$, $\mu$ is a probability measure. Hence it is enough to show that $Supp(\mu) = \{x\}$. Since $\mu$ is regular, it suffices to show that for every compact subset K of $X_\omega - \{x\}$, $\mu(K) = 0$.

Let K be a compact subset of $X_\omega - \{x\}$ and $y \in K$, then $y \neq x$. Since S separates points of X, there exists $h \in S$ such that $h(y) \neq h(x)$. Therefore there is a neighborhood $U_y$ of y such that $h(z) \neq h(x)\ \forall\ z \in U_y$. Now $\left\{U_y : y \in K\right\}$ is an open cover of K. Since K is compact, there exist finitely many function $h_1,\ldots,h_n \in S$ such that for each $y \in K$, $h_i(y) \neq h_i(x)$ for some i, $1 \le i \le n$.

Let $g = \sum_{i=1}^{n} |h_i' - h_i'(x)|^2$, where $h_i'$ denotes the continuous extension of $h_i$ to $X_\omega$. It is clear that g is nonnegative and $g > 0$ on K.

$$\text{Now} \int_{X_\omega} g\, d\mu = \sum_{i=1}^{n} \int_{X_\omega} |h_i'|^2\, d\mu - \sum_{i=1}^{n} h_i'(x) \int_{X_\omega} h_i'\, d\mu$$

$$- \sum_{i=1}^{n} h_i'(x) \int_{X_\omega} \overline{h_i'}\, d\mu + \sum_{i=1}^{n} |h_i'(x)|^2 \int_{X_\omega} 1\, d\mu$$

$$= \sum_{i=1}^{n} |h_i'(x)|^2 - 2 \sum_{i=1}^{n} |h_i'(x)|^2 + \sum_{i=1}^{n} |h_i'(x)|^2$$

(since $\mu = \delta_x$ on H)

$$= 0.$$

Hence $g = 0$ $\mu$-almost everywhere. But $g > 0$ on K, therefore, $\mu(K) = 0$. This completes the proof.

The proof of Proposition 3.1.2 implies the following corollary.

Corollary 3.1.4 Let $\{f_1, \ldots, f_n\}$ be a finite subset of $C_o(X,\mathbb{C})$ separating points of X and let f be a strictly positive function in $C_o(X,\mathbb{R})$. Then $\left\{f, f\, f_1, \ldots, f\, f_n, f \sum_{i=1}^{n} |f_i|^2\right\}$ is a universal Korovkin set in $C_o(X,\mathbb{C})$ w.r.t. positive operators.

Corollary 3.1.2, now combined with Proposition 3.1.2 and Corollary 3.1.4, yields the following:

Corollary 3.1.5 Let A be a commutative Banach algebra with continuous symmetric involution and satisfying property 'P'.

Suppose that A contains a set S and an element x such that $\hat{S}$ separates points of $\Delta A$ and $\hat{x}$ is strictly positive on $\Delta A$. Then $\{x\} \cup x\, S \cup x\, SS^*$ is a universal Korovkin set in A w.r.t. positive operators.

If S is finite, say $S = \left\{x_1,\ldots,x_n\right\}$, then $\left\{x,\ xx_1,\ldots,xx_n,\ x\sum_{i=1}^{n} x_i x_i^*\right\}$ is a universal Korovkin set in A w.r.t. positive operators.

Remark 3.1.1 Corollary 3.1.5 indicates a way for constructing universal Korovkin sets w.r.t. positive operators. We shall use it to provide some examples at the end of this section.

Now we are ready to prove Theorem 3.1.1.

Proof <u>of</u> Theorem 3.1.1 (i) ==> (ii) is obvious. In fact, members of finite universal Korovkin set in A will satisfy the requirements of (ii).

Conversely, let $S = \left\{x_1,\ldots,x_n\right\}$ be a finite subset of A such that $\hat{S}$ separates points of $\Delta A$. In view of Corollary 3.1.5, it suffices to show that there exists an element $x \in A$ such that $\hat{x} > 0$. For this first we show that $\Delta A$ is second countable.

Let $\Delta_\omega A = \Delta A \cup \{0\}$ be the one point compactification of $\Delta A$. Let $\hat{x}_i'$ denotes the continuous extension of the function $\hat{x}_i$ to $\Delta_\omega A$. Consider the map

$$\Phi : \Delta_\omega A \longrightarrow \mathbb{C}^{n+1} \text{ defined as}$$

$$\Phi(\gamma) = (1,\ \hat{x}_1'(\gamma),\ldots,\hat{x}_n'(\gamma)), \qquad \gamma \in \Delta_\omega A.$$

Since the functions 1, $\hat{x}_1', \ldots, \hat{x}_n'$ separate points of $\Delta_\omega A$, $\Phi$ is one to one. Also $\Phi$ is continuous. Since $\Delta_\omega A$ is compact and $\mathbb{C}^{n+1}$ is Hausdorff, $\Phi$ is a homeomorphism of $\Delta_\omega A$ onto $\Phi(\Delta_\omega A)$. Thus $\Delta_\omega A$ may be regarded as a compact subset of $\mathbb{C}^{n+1}$. Therefore $\Delta A$ is second countable.

Now symmetry of involution in A implies that $m(x^* x) \geq 0 \ \forall \ m \in \Delta A$ and $\forall \ x \in A$. Further, since $\Delta A$ consists of nontrivial homomorphisms;

$\forall \ m \in \Delta A$, there exists $x_m \in A$ (depending on m) such that $m(x_m) \neq 0$. Hence $m(x_m^* x_m) > 0$ or equivalently $(x_m^* x_m)\hat{}\,(m) > 0$.

Now continuity of Gelfand transform implies that there is a neighborhood $U_m$ of m in $\Delta A$ such that $(x_m^* x_m)\hat{}\,(\eta) > 0 \ \forall \ \eta \in U_m$. Thus $\left\{U_m : m \in \Delta A\right\}$ is an open cover of $\Delta A$. Since $\Delta A$ is second countable there is a countable subcover $\left\{U_{m_j}\right\}_{j=1}^{\infty}$. For each j, let us write $y_j$ for $x_{m_j}^* \, x_{m_j}$. Now let $x = \sum_{j=1}^{\infty} \frac{y_j}{2^j \|y_j\|}$, it is clear that $x \in A$ and $\hat{x} > 0$.

This completes the proof.

Remark 3.1.2 Proof of Theorem 3.1.1 reveals the following fact, which may be of independent interest:

Let A be a commutative Banach algebra with symmetric involution. Then following are equivalent.

(i) Finitely many elements in $\hat{A}$ separate points of $\Delta A$.

(ii) Finitely many elements in $\hat{A}$ separate points of $\Delta A$ strongly.

Furthermore, following fact, which we shall refer frequently in the sequel, is observed in the proof of Theorem 3.1.1.

Let A be a commutative Banach algebra. If finitely many elements in $\hat{A}$ separate points of $\Delta A$ then $\Delta A$ is a finite dimensional separable metric space.

Remark 3.1.3 In [45, p. 452], Pannenberg has observed the following fact:

Let A be a commutative Banach algebra with continuous symmetric involution and bounded approximate identity. Then following are equivalent.

(1) A admits a finite universal Korovkin set w.r.t. positive contraction operators.

(2) There exist finitely many elements in A such that their Gelfand transforms separate strongly the points of $\Delta A$.

In view of Remark 3.1.2 now the word 'strongly' may be omitted in (2).

An immediate corollary of Theorem 3.1.1 is the following :

Corollary 3.1.6 Let A be a commutative Banach algebra with continuous symmetric involution and satisfying property 'P'. Then following are equivalent.

(i) A admits a finite universal Korovkin set w.r.t. positive operators.

(ii) A admits a finite universal Korovkin set w.r.t. positive contraction operators.

(iii) There exist finitely many elements in A such that their Gelfand transforms separate points of $\Delta A$.

In view of Corollary 3.1.6, results w.r.t. positive contraction operators obtained by Pannenberg in [45] hold w.r.t. positive operators as well. For example, Theorem 1 and Proposition 3 of [45] give rise to following corollaries.

Corollary 3.1.7 For a locally compact abelian group G with dual group $\hat{G}$, following are equivalent.

(i) $L^1(G)$ admits a finite universal Korovkin set w.r.t. positive operators.

(ii) $\hat{G}$ is finite dimensional separable metric space.

Corollary 3.1.8 Let A, B be commutative Banach algebras with continuous symmetric involution and bounded approximate identity. Let $A \hat{\otimes}_{\pi} B$ be the projective tensor product of A and B. Then $A \hat{\otimes}_{\pi} B$ admits a finite universal Korovkin set w.r.t. positive operators iff A and B admit.

We conclude this section with some examples of application of Corollary 3.1.4 and Corollary 3.1.5.

Example 3.1.1 Consider the Banach algebra $C_o(\mathbb{R}^n,\mathbb{C})$, $n \geq 1$.

Let $S = \left\{ x_1 \exp\ (-\|x\|^2),\ldots,x_n \exp\ (-\|x\|^2) \right\}$ and $f(x) = \exp(-\|x\|^2)$, where $\|x\|^2 = \sum_{i=1}^{n} |x_i|^2$ for $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$. It is clear that S separates points of $\mathbb{R}^n$ and $f > 0$. An application of Corollary 3.1.4 yields that

$\left\{\exp(-\|x\|^2),\ x_1 \exp(-2\|x\|^2), \ldots, x_n \exp(-2\|x\|^2),\ \|x\|^2 \exp(-3\|x\|^2)\right\}$

is a universal Korovkin set in $C_o(\mathbb{R}^n,\mathbb{C})$ w.r.t. positive operators. In particular, if $n = 1$, $\left\{\exp(-x^2),\ x \exp(-2x^2),\ x^2 \exp(-3x^2)\right\}$ is a universal Korovkin set in $C_o(\mathbb{R},\mathbb{C})$ w.r.t. positive operators. Note that this set consists of three functions. The next example shows that $C_o(\mathbb{R},\mathbb{C})$ has a universal Korovkin set consisting of just two functions.

Example 3.1.2 Let us consider the Banach algebra $L^1(\mathbb{R})$ with involution ~, where

$$f^{\sim}(x) = f(-x).$$

Let $\lambda$ and $\mu$ be two fixed real numbers such that $\lambda > 0$. We define functions f and g in $L^1(\mathbb{R})$ as

$$f(x) = \begin{cases} \exp[(\lambda+i\mu)x] & \text{if } x < 0 \\ \exp[(-\lambda+i\mu)x] & \text{if } x \geq 0 \end{cases}$$

and

$$g(x) = \begin{cases} 0 & \text{if } x < 0 \\ \exp[(-\lambda+i\mu)x] & \text{if } x \geq 0 \end{cases} .$$

A simple computation shows that

$f = g+\tilde{g} = 2\lambda\, g * \tilde{g}$, where * denotes the convolution in $L^1(\mathbb{R})$,

$$\text{and} \qquad \hat{g}(x) = \frac{i}{(\mu-x) + i\lambda} .$$

It is clear that $\hat{g}$ separates points of $\hat{\mathbb{R}}$ and $\hat{f} = 2\lambda|\hat{g}|^2 > 0$. Now, Corollary 3.1.5 applied to this situation gives

$$L^1(\mathbb{R}) = \mathrm{Kor}_u^+\left(\left\{f, f * g,\ f * g * \tilde{g}\right\}\right)$$

$$= \mathrm{Kor}_u^+\left(\left\{f, (g+\tilde{g}) * g,\ (g+\tilde{g}) * (g+\tilde{g})\ \frac{1}{2\lambda}\right\}\right)$$

$$= \mathrm{Kor}_u^+\left(\left\{f,\ g * g + \tilde{g} * g,\ g * g + \tilde{g} * \tilde{g} + 2g * \tilde{g}\right\}\right)$$

(we can neglect scalar multiples)

$$= \mathrm{Kor}_u^+\left(\left\{f,\ g * g + \frac{1}{2\lambda} f,\ g * g + (g * g)^{\sim} + \frac{1}{\lambda} f\right\}\right)$$

$$= \mathrm{Kor}_u^+\left(\left\{f,\ g * g\right\}\right)$$

$$= \mathrm{Kor}_u^+\left(\left\{f, h\right\}\right),$$

where $h(x) = g * g(x) = \begin{cases} 0 & \text{if } x < 0 \\ x \exp[(-\lambda+i\mu)x] & \text{if } x \geq 0 \end{cases}$

Thus $\left\{f, h\right\}$ is a universal Korovkin set w.r.t. positive operators in $L^1(\mathbb{R})$ consisting of just two functions.

Observe that in view of Corollary 3.1.2, $\left\{\hat{f}, \hat{h}\right\}$ is a universal Korovkin set in $C_o(\mathbb{R},\mathbb{C})$ w.r.t. positive operators. Therefore, the set of functions $\left\{\frac{1}{(\mu-x)^2 + \lambda^2},\ \frac{1}{[(\mu-x) + i\lambda]^2}\right\}$ is a universal Korovkin set in $C_o(\mathbb{R},\mathbb{C})$.

Example 3.1.3 Let G be any compact metrizable abelian group. Since G is metrizable, its dual group $\hat{G}$ is countable. Let $\{\gamma_n\}_{n=1}^{\infty}$ be an enumeration of $\hat{G}$. Now consider the function

$f = \sum_{n=1}^{\infty} \frac{1}{2^n} \gamma_n$. It is clear that $\hat{f} > 0$ and $\hat{f}$ separates points of $\hat{G}$. By Corollary 3.1.5, $\left\{ f,\ f * f,\ f * f * \tilde{f} \right\}$ is a universal Korovkin set in $L^1(G)$ w.r.t. positive operators. More explicitly, this set is $\left\{ \sum_{n=1}^{\infty} \frac{1}{2^n} \gamma_n,\ \sum_{n=1}^{\infty} \frac{1}{4^n} \gamma_n,\ \sum_{n=1}^{\infty} \frac{1}{8^n} \gamma_n \right\}$.

In fact, if $\{\alpha_n\}_{n=1}^{\infty}$ is any sequence of distinct positive numbers satisfying $\sum_{n=1}^{\infty} \alpha_n < \infty$, then $\left\{ \sum_{n=1}^{\infty} \alpha_n \gamma_n,\ \sum_{n=1}^{\infty} \alpha_n^2 \gamma_n,\ \sum_{n=1}^{\infty} \alpha_n^3 \gamma_n \right\}$ is a universal Korovkin set in $L^1(G)$ w.r.t. positive operators.

This example shows that for any compact metrizable abelian group G, the algebra $L^1(G)$ admits a universal Korovkin set w.r.t. positive operators consisting of three functions.

Remark 3.1.4 In the above example the function f is such that $\hat{f}$ separates points of $\hat{G}$ strongly. Therefore, Theorem 2.6.9 implies that $\left\{ f,\ f * \tilde{f} \right\}$ is a universal Korovkin set in $L^1(G)$ w.r.t. positive contraction operators. We now show that this set fails to be a universal Korovkin set in $L^1(G)$ w.r.t. positive operators. Consider the function $g = \frac{1}{64} \gamma_1 + \frac{29}{32} \gamma_2 + \frac{1}{8} \gamma_3$.

It is clear that g is a continuous positive definite function on G. Thus g defines a continuous positive linear functional on $L^1(G)$. Also,

$$< f, g > = \frac{1}{2^2} = < f, \gamma_2 >$$

and

$$< f * \tilde{f}, g > = \frac{1}{4^2} = < f * \tilde{f}, \gamma_2 >.$$

But $g \neq \gamma_2$. Hence, in view of Theorem 2.6.5, $\left\{ f,\ f * \tilde{f} \right\}$ is not a

universal Korovkin set in $L^1(G)$ w.r.t. positive operators.

§2.

Let G be a locally compact group. Consider $Z(L^1(G))$, centre of the group algebra $L^1(G)$. It is well known that $Z(L^1(G))$ is nontrivial iff G is an [IN] group. Further, it is well established that $Z(L^1(G))$ is a commutative Banach algebra with bounded approximate identity, see, §2.4.

Motivated by the results obtained by Pannenberg [45] and also by Corollary 3.1.7 of last section, one is tempted to ask the following:

Question 3.2.1 When does $Z(L^1(G))$ admit a finite universal Korovkin set w.r.t. positive operators?

In this section we provide complete answer to this question for the case when G is a compact group and for the case when G is a connected [Z] group.

Let G be a compact group and $\hat{G}$ be its dual object. It is well known that the maximal ideal space of $Z(L^1(G))$ can be identified with the set of all normalized characters of G, with discrete topology, see ([26], §28.71, p. 109). We denote this set by $\mathfrak{X} = \left\{ \frac{\chi_\sigma}{d_\sigma} : \sigma \in \hat{G} \right\}$. If $\tau_\sigma$ denote the multiplicative linear functional corresponding to $\sigma \in \hat{G}$, then

$$\tau_\sigma(f) = \int_G f(x) \frac{\chi_\sigma(x)}{d_\sigma} dx, \qquad f \in Z(L^1(G)).$$

We shall show that if $G$ is a compact group then $Z(L^1(G))$ has a finite universal Korovkin set iff $G$ is metrizable. More precisely, we have the following:

Proposition 3.2.1 Let $G$ be a compact group. Then $Z(L^1(G))$ admits a finite (three element) universal Korovkin set w.r.t. positive operators iff $G$ is metrizable.

Proof Suppose $Z(L^1(G))$ admits a finite universal Korovkin set. This means that there are finitely many functions in $Z(L^1(G))$ such their Gelfand transforms separate points of $\mathcal{X} = \left\{ \frac{\chi_\sigma}{d_\sigma} : \sigma \in \hat{G} \right\}$. In view of Remark 3.1.2, $\mathcal{X}$ is second countable, hence so is $\hat{G}$. Now since $\hat{G}$ is discrete, it follows that $\hat{G}$ is countable. Therefore, by Corollary 28.11 ([26], p. 65), $\hat{G}$ is metrizable.

Conversely, suppose that $G$ is metrizable. Then $\hat{G}$ is countable. We shall show that there exists a function in $Z(L^1(G))$ whose Gelfand transform is strictly positive as well as separates points of $\mathcal{X}$.

Let $\left\{\sigma_n\right\}_{n=1}^{\infty}$ be an enumeration of $\hat{G}$. Let $\left\{\alpha_n\right\}_{n=1}^{\infty}$ be a sequence of distinct positive numbers satisfying $\sum\limits_{n=1}^{\infty} \alpha_n d_{\sigma_n}^2 < \infty$.

Define $f(x) = \sum\limits_{n=1}^{\infty} \alpha_n d_{\sigma_n} \chi_{\sigma_n}(x)$, $x \in G$. It is clear that $f \in Z(L^1(G))$.

Now, for any positive integer $m$,

$$\tau_{\sigma_m}(f) = \int_G f(x) \frac{\chi_{\sigma_m}(x)}{d_{\sigma_m}} dx = \frac{1}{d_{\sigma_m}} \int_G \sum_{n=1}^{\infty} \alpha_n d_{\sigma_n} \chi_{\sigma_n}(x) \chi_{\sigma_m}(x) dx$$

$$= \frac{1}{d_{\sigma_m}} \sum_{n=1}^{\infty} \alpha_n \, d_{\sigma_n} \int_G \chi_{\sigma_n}(x) \, \chi_{\sigma_m}(x) \, dx$$

$= \alpha_m$, in view of orthogonality relation between characters.

Since $\alpha_n^{'s}$ are positive numbers, $\tau_\sigma(f) > 0 \ \forall \ \sigma \in \hat{G}$.

Now suppose that $\tau_{\sigma_m}(f) = \tau_{\sigma_k}(f)$, then we get $\alpha_m = \alpha_k$. Since $\alpha_n^{'s}$ are distinct, it turns out that $\sigma_m = \sigma_k$. Thus points of $\mathcal{X}$ are separated by Gelfand transform of f.

By Corollary 3.1.5, $\left\{f,\ f * f,\ f * f * \tilde{f}\right\}$ is a universal Korovkin set in $Z(L^1(G))$ w.r.t. positive operators. More explicitly, this set is

$$\left\{ \sum_{n=1}^{\infty} \alpha_n \, d_{\sigma_n} \, \chi_{\sigma_n},\ \sum_{n=1}^{\infty} \alpha_n^2 \, d_{\sigma_n} \, \chi_{\sigma_n},\ \sum_{n=1}^{\infty} \alpha_n^3 \, d_{\sigma_n} \, \chi_{\sigma_n} \right\}.$$

Remark 3.2.1 In view of Theorem 2.6.9, if f is the function constructed in the proof of Proposition 3.2.1 then $\left\{f,\ f * \tilde{f}\right\} = \left\{ \sum_{n=1}^{\infty} \alpha_n \, d_{\sigma_n} \, \chi_{\sigma_n},\ \sum_{n=1}^{\infty} \alpha_n^2 \, d_{\sigma_n} \, \chi_{\sigma_n} \right\}$ is a universal Korovkin set in $Z(L^1(G))$ w.r.t. positive contraction operators.

Next we establish a proposition and some corollaries which will lead us to an answer of Question 3.2.1 when G is a connected [Z] group. In the remaining part of this chapter, unless otherwise stated, G will denote a [Z] group and Z (or Z(G)) will denote centre of G, $d\dot{x}$ will denote the Haar measure on G/Z.

We shall need the following result due to Grosser and Moskowitz ([23], Theorem 2.1).

Lemma 3.2.1 Let $G$ be a [Z] group. Define the operator $\#$ on $L^1(G)$ as

$$f^{\#}(y) = \int_{G/Z} f(xyx^{-1})\, d\dot{x}\ , \quad y \in G. \quad \text{Then } Z(L^1(G)) = (L^1(G))^{\#}.$$

Proposition 3.2.2 Let $G_1$ and $G_2$ be [Z] groups. Then $Z(L^1(G_1 \times G_2)) = Z(L^1(G_1)) \hat{\otimes}_{\pi} Z(L^1(G_2))$, where $\hat{\otimes}_{\pi}$ denotes the projective tensor product.

Proof It is well known that $L^1(G_1 \times G_2) = L^1(G_1) \hat{\otimes}_{\pi} L^1(G_2)$. In view of this relation and Lemma 3.2.1, it suffices to show that

$$(f_1 \otimes f_2)^{\#} = f_1^{\#} \otimes f_2^{\#} \quad \forall\ f_1 \in L^1(G_1) \text{ and } \forall\ f_2 \in L^1(G_2).$$

Observe that $Z(G_1 \times G_2) = Z(G_1) \times Z(G_2)$. Hence Second Isomorphism Theorem for topological groups implies that

$$(G_1 \times G_2)/Z(G_1 \times G_2) \simeq G_1/Z(G_1) \times G_2/Z(G_2).$$

Let $d\dot{x}$ and $d\dot{t}$ denote normalized Haar measures on $G_1/Z(G_1)$ and $G_2/Z(G_2)$ respectively, then the product measure $d(\dot{x},\dot{t})$ will be normalized Haar measure on

$$G_1/Z(G_1) \times G_2/Z(G_2) = (G_1 \times G_2)/Z(G_1 \times G_2).$$

Now for $(y_1,y_2) \in G_1 \times G_2$,

$$(f_1 \otimes f_2)^{\#}(y_1,y_2) = \int_{(G_1 \times G_2)/Z(G_1 \times G_2)} (f_1 \otimes f_2)((x,t)(y_1,y_2)(x,t)^{-1})\, d(\dot{x},\dot{t})$$

$$= \int_{G_1/Z(G_1) \times G_2/Z(G_2)} (f_1 \otimes f_2)\ (xy_1x^{-1}, ty_2t^{-1})\ d(\dot{x},\dot{t})$$

$$= \int_{G_1/Z(G_1)} \int_{G_2/Z(G_2)} f_1(xy_1x^{-1}) f_2(ty_2t^{-1}) d\dot{x}\, d\dot{t}$$

$$= \int_{G_1/Z(G_1)} f_1(xy_1x^{-1})\, d\dot{x} \int_{G_2/Z(G_2)} f_2(ty_2t^{-1}) d\dot{t}$$

$$= f_1^{\#}(y_1)\, f_2^{\#}(y_2) = (f_1^{\#} \otimes f_2^{\#})\, (y_1, y_2).$$

This completes the proof.

Proposition 3.2.2 combined with Corollary 3.1.8 yields the following;

Corollary 3.2.1 Let $G_1$ and $G_2$ be [Z] groups. Then $Z(L^1(G_1 \times G_2))$ admits a finite universal Korovkin set <==> $Z(L^1(G_1))$ as well as $Z(L^1(G_2))$ do.

If $G = V \times K$, where V is a vector group and K is a compact group, we get following answer to the Question 3.2.1.

Corollalry 3.2.2 Let $G = V \times K$. Then $Z(L^1(G))$ admits a finite universal Korovkin set iff G is metrizable.

Proof Suppose $Z(L^1(G))$ has a finite universal Korovkin set. Then, Corollary 3.2.1 implies that $Z(L^1(K))$ admits a finite universal Korovkin set. Hence K is metrizable (Proposition 3.2.1). Since V is a vector group, it follows that G is metrizable.

Conversely, suppose G is metrizable. This means that K is metrizable. Therefore by Proposition 3.2.1, $Z(L^1(K))$ admits a finite universal Korovkin set. Since $Z(L^1(V)) = L^1(V)$ admits a

finite universal Korovkin set, it follows that $Z(L^1(G))$ admits a finite universal Korovkin set.

Corollary 3.2.2 already contains an answer to the Question 3.2.1 for some special classes of [IN] groups. Recall that we have the relationship

$$[Z] \subsetneq [FIA]^- \subsetneq [SIN] \subsetneq [IN].$$

Corollary 3.2.3 Let G be a group satisfying either of the following two conditions,

(i) G is a connected [SIN] group.

(ii) G is an $[FIA]^-$ group such that $G/G_o$ is compact, where $G_o$ denotes the connected component of identity in G.

Then $Z((L^1(G))$ admits a finite universal Korovkin set iff G is metrizable.

Proof Observe that if G satisfies either of the two conditions, then $G = V \times K$, where V is a vector group and K is a compact group, see, Grosser and Moskowitz ([21], Theorem 4.3, p. 326) and ([24], Proposition 4.4, p. 26) respectively. The corollary now is an immediate consequence of the Corollary 3.2.2.

## §3

In this section we continue to investigate the Question 3.2.1 for the case when G is a [Z] group. In Corollary 3.2.3, we have obtained complete answer to the Question for the case when G is a connected [Z] group or more generally for the [Z] groups G

satisfying $G/G_o$ is compact, where $G_o$ is the connected component of identity in G.

By structure theorem ([21], Theorem 4.4, p. 328) every [Z] group G has the form $G = V \times H$, where V is a vector group and H contains a compact open normal subgroup K. In view of this, Proposition 3.2.2 and Corollary 3.2.1 we need investigate only those [Z] groups which contain a compact open normal subgroup. Note that if a [Z] group G contains a compact open normal subgroup then it contains such a subgroup K such that G/K is abelian, see ([21], Corollalry 2, p. 331).

It is well known that every continuous irreducible unitary representation of a [Z] group is finite dimensional, ([22], Theorem 2.1). As usual $\hat{G}$ will denote the set of all equivalence classes of continuous irreducible unitary representations of G. For $\sigma \in \hat{G}$, $\chi_\sigma$ and $d_\sigma$ will denote its character and dimension respectively. If $\sigma \in \hat{G}$, we define

$$\tau_\sigma(f) = \int_G f(x) \frac{\bar{\chi}_\sigma(x)}{d_\sigma} dx \quad \text{for } f \in Z(L^1(G)).$$

Then $\tau_\sigma$ defines a continuous multiplicative functional on $Z(L^1(G))$. It is shown in Section 6 of [23] that every continuous multiplicative functional on $Z(L^1(G))$ arises in this manner and the maximal ideal space of $Z(L^1(G))$ can be identified with the set $\mathcal{X} = \left\{\frac{\chi_\sigma}{d_\sigma} : \sigma \in \hat{G}\right\}$ with the topology of uniform convergence on compact subsets of G. Topology on $\mathcal{X}$ gives rise to a topology on $\hat{G}$ in a natural way.

We begin with an investigation of the Question 3.2.1 for [Z] groups $G$ containing a compact open normal subgroup $K$ such that $G = KZ$. If $G$ is such a group then it can be seen easily that $G/K$ is abelian. We shall prove the following theorem.

**Theorem 3.3.1** Let $G$ be a [Z] group. Suppose that it has a compact open normal subgroup $K$ such that $G = KZ$. Further suppose that (i) $G$ is a separable metric space and (ii) $G/K$ has finite torsion free rank. Then $Z(L^1(G))$ admits a finite universal Korovkin set.

First we establish a Lemma which is needed for the proof of Theorem 3.3.1.

**Lemma 3.3.1** Let $G$ be a [Z] group such that $G = KZ$ for a compact subgroup $K$ of $G$. Then $\hat{K} = \left\{\sigma|_K : \sigma \in \hat{G}\right\}$.

**Proof** Let $\sigma \in \hat{G}$. For $x \in G$ and $z \in Z$ we have

$$\sigma_x \sigma_z = \sigma_{xz} = \sigma_{zx} = \sigma_z \sigma_x.$$

Since $\left\{\sigma_x : x \in G\right\}$ is an irreducible set of operators, it follows that $\sigma_z = \lambda_\sigma(z)\, I_\sigma$, where $I_\sigma$ is the identity operator on $H_\sigma$ (the representation space of $\sigma$) and $\lambda_\sigma(z)$ is a scalar depending on $\sigma$ and $z$. It is easy to see that $\lambda_\sigma \in \hat{Z}$. Thus $\sigma_{xz} = \lambda_\sigma(z)\, \sigma_x$ $\forall\, x \in G$ and $\forall\, z \in Z$.

We shall prove that $\sigma|_K$ is irreducible.

Let $T$ be an operator on $H_\sigma$ such that $T\,\sigma_k = \sigma_k T$ $\forall\, k \in K$. Let $x \in G$ then $x = kz$ for some $k \in K$ and $z \in Z$.

Now, $T\,\sigma_x = T\,\sigma_{kz} = \lambda_\sigma(z)\; T\,\sigma_k$

and $\sigma_x T = \sigma_{kz} T = \lambda_\sigma(z)\, \sigma_k T$.

Therefore, we have $T \sigma_x = \sigma_x T \; \forall \; x \in G$. But since $\left\{\sigma_x : x \in G\right\}$ is an irreducible set of operators, it follows that T is a scalar multiple of $I_\sigma$. This proves that $\sigma|_K$ is irreducible. Thus $\hat{K} \supseteq \left\{\sigma|_K : \sigma \in \hat{G}\right\}$.

In Theorem 5.1 of [22], Grosser and Moskowitz have shown that if K is a compact subgroup of a [Z] group G and $\sigma \in \hat{K}$, then there exists a $\rho \in \hat{G}$ such that $\sigma$ is unitarily equivalent to an irreducible component of $\rho|_K$. The inclusion $\hat{K} \subseteq \left\{\sigma|_K : \sigma \in \hat{G}\right\}$ now follows from this result.

Proof of Theorem 3.3.1 In view of Theorem 3.1.1, it suffices to show that there are finitely many functions in $Z(L^1(G))$ such that their Gelfand transforms separate points of $\mathfrak{X}$

Let r be the (finite) torsion free rank of G/K. This means that there exist $x_1,\ldots,x_r \in G$ such that set of corresponding cosets, $\left\{x_1K,\ldots,x_rK\right\}$ is a maximal independent set of elements of infinite order in G/K. Moreover, since G = KZ, $x_1,\ldots,x_r$ may be chosen to be central elements of G.

Let H be the subgroup generated by K and $x_1,x_2,\ldots,x_r$. Then H is an open normal subgroup of G and hence G/H is discrete. Since $K \subset H$ and G/K is abelian, G/H is abelian. Since G is separable metric space, G/H is second countable and therefore G/H is countable. Moreover, G/H is a torsion group since $\left\{x_1K,\ldots,x_rK\right\}$ is a maximal independent set of elements of infinite order in G/K.

Since $G/H$ is countable and $G = HZ$, we may choose representatives $\{s_j\}_{j\in J}$ of members of $G/H$ such that each $s_j$ is a central element of $G$, where $J$ is a countable index set. Since $G/H$ is a torsion group $\{\gamma(s_jH) : \gamma \in (G/H)^\wedge\}$ is a finite subgroup of the circle $\mathbb{T}$ for each $j \in J$. Therefore, for each $j \in J$, there exists $\epsilon_j > 0$ such that

$$|\gamma(s_jH) - 1| \geq \epsilon_j \ \forall\ \gamma \in (G/H)^\wedge \text{ such that } \gamma(s_jH) \neq 1. \qquad (3.3.1)(a)$$

Note that $\epsilon_j$ depends only on the coset $s_jH$. Further, we may assume that $\epsilon_j < 1\ \forall\ j \in J$.

Let $\{\delta_j\}_{j\in J}$ be a summable sequence of positive numbers satisfying

$$\epsilon_n\delta_n - 2\sum_{j\geq n+1} \delta_j > 0 \qquad \forall\ n \in J. \qquad (3.3.1(b)$$

(One such sequence is obtained by defining $\delta_j = \frac{1}{4^j}\prod_{k=1}^{j-1} \epsilon_k$, see [45, §3, p. 453]).

Now, $K$ is a compact open normal subgroup of $G$ such that $G = KZ$. Therefore, it follows from Lemma 3.3.1 that $\hat{K} = \{\sigma|_K : \sigma \in \hat{G}\}$. Moreover, $G$ is metrizable. Therefore $K$ is metrizable and $\hat{K}$ is countable. Thus we may choose a sequence $\{\sigma_n\}_{n=1}^{\infty}$ in $\hat{G}$ such that $\hat{K} = \{\sigma_n|_K\}_{n=1}^{\infty}$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of distinct positive numbers satisfying $\sum_{n=1}^{\infty} \alpha_n\, d_{\sigma_n}^2 < \infty$.

Now we define r+2 functions on $G$ as follows:

$$F_o = \sum_{n=1}^{\infty} \alpha_n\, d_{\sigma_n}\, \chi_{\sigma_n}\, \xi_K,$$

($\xi_K$ denotes the characteristic function of K)

$F_i(x) = F_o(xx_i^{-1})$ for $1 \le i \le r$ and

$F_{r+1}(x) = \sum_{j\in J} \delta_j \, F_o(xs_j^{-1})$.

Since each $\chi_{\sigma_n}$ is a continuous central function and $\sum_{n=1}^{\infty} \alpha_n \, d_{\sigma_n}^2 < \infty$ and K is a compact normal subgroup, $F_o$ is a continuous central function with compact support. Thus $F_o$ is in $Z(L^1(G))$. Further, since $Z(L^1(G))$ is closed under translation $F_1, F_2, \ldots, F_r$ belong to $Z(L^1(G))$. Also, since $\sum_{j\in J} \delta_j < \infty$, $F_{r+1}$ belongs to $Z(L^1(G))$.

We shall prove that the Gelfand transforms of the functions $F_o, F_1, \ldots F_r, F_{r+1}$ separate point of $\mathfrak{X}$.

Suppose $\sigma, \rho \in \hat{G}$ such that $\tau_\sigma(F_i) = \tau_\rho(F_i)$ $\forall$ $i = 0,1,\ldots,r,r+1$. We must show that $\sigma = \rho$. Let dx denotes the Haar measure on G normalized so that measure of K is equal to 1.

Since $\sigma|_K \in \hat{K}$, we have $\sigma|_K = \sigma_m|_K$ for some $m \in \mathbb{N}$. Also $d_\sigma = d_{\sigma_m}$.

Now,

$$\tau_\sigma(F_o) = \frac{1}{d_\sigma} \int_G \left( \sum_{n=1}^{\infty} \alpha_n d_{\sigma_n} \chi_{\sigma_n} \xi_K \right)(x) \, \chi_\sigma(x) \, dx$$

$$= \frac{1}{d_\sigma} \int_K \sum_{n=1}^{\infty} \alpha_n d_{\sigma_n} \chi_{\sigma_n}(x) \, \chi_\sigma(x) \, dx$$

$$= \frac{1}{d_{\sigma_m}} \sum_{n=1}^{\infty} \alpha_n d_{\sigma_n} \int_K \chi_{\sigma_n}(x) \, \overline{\chi_{\sigma_m}(x)} \, dx$$

$$= \alpha_m.$$

We note that $\tau_\sigma(F_o) > 0 \quad \forall\ \sigma \in \hat{G}$.

Suppose $\rho|_K = \sigma_l|_K$ for some $l \in \mathbb{N}$. Then $\tau_\rho(F_o) = \alpha_l$. Therefore, if $\tau_\sigma(F_o) = \tau_\rho(F_o)$ then $\sigma|_K = \rho|_K$, as $\alpha_n$'s are distinct.

Next we compute $\tau_\sigma(F_i)$ for $i \le i \le r$.

$$\tau_\sigma(F_i) = \frac{1}{d_\sigma} \int_G \sum_{n=1}^{\infty} \alpha_n d_{\sigma_n} \chi_{\sigma_n} (xx_i^{-1})\ \xi_K(xx_i^{-1})\ \chi_\sigma(x)\ dx$$

$$= \frac{1}{d_\sigma} \int_G \sum_{n=1}^{\infty} \alpha_n d_{\sigma_n} \chi_{\sigma_n} (x)\ \xi_K(x)\ \chi_\sigma(xx_i)\ dx$$

(by invariance of Haar integral)

$$= \frac{1}{d_\sigma} \int_K \sum_{n=1}^{\infty} \alpha_n d_{\sigma_n} \chi_{\sigma_n}(x)\ \chi_\sigma(xx_i)\ dx$$

$$= \frac{1}{d_\sigma} \sum_{n=1}^{\infty} \alpha_n d_{\sigma_n} \int_K \chi_{\sigma_n}(x)\ \chi_\sigma(xx_i)\ dx.$$

Now,

$$\int_K \chi_{\sigma_n}(x)\ \chi_\sigma(xx_i)\ dx = \int_K \sum_{l=1}^{d_{\sigma_n}} u_{ll}^{(\sigma_n)}(x) \sum_{r=1}^{d_\sigma} u_{rr}^{(\sigma)}(xx_i)\ dx$$

$$= \int_K \sum_{l=1}^{d_{\sigma_n}} u_{ll}^{(\sigma_n)}(x) \sum_{r=1}^{d_{\sigma_m}} \sum_{j=1}^{d_{\sigma_m}} u_{rj}^{(\sigma_m)}(x)\ u_{jr}^{(\sigma)}(x_i)\ dx$$

(note that $\sigma|_K = \sigma_m|_K$)

$$= \sum_{l=1}^{d_{\sigma_n}} \sum_{r=1}^{d_{\sigma_m}} \sum_{j=1}^{d_{\sigma_m}} \int_K u_{ll}^{(\sigma_n)}(x)\ u_{rj}^{(\sigma_m)}(x)\ dx\ u_{jr}^{(\sigma)}(x_i)$$

$$= \sum_{l=1}^{d_{\sigma_n}} \sum_{r=1}^{d_{\sigma_m}} \sum_{j=1}^{d_{\sigma_m}} \frac{1}{d_{\sigma_m}} \delta_{lr}\, \delta_{lj}\, \delta_{mn}\, u_{jr}^{(\sigma)}(x_i)$$

$$= \begin{cases} 0 & \text{if } n \neq m \\ \dfrac{1}{d_{\sigma_m}} \chi_\sigma(x_i) & \text{if } n = m\,. \end{cases}$$

Thus,

$$\tau_\sigma(F_i) = \frac{1}{d_{\sigma_m}} \alpha_m\, d_{\sigma_m} \frac{1}{d_{\sigma_m}} \chi_\sigma(x_i) = \frac{\alpha_m}{d_{\sigma_m}} \chi_\sigma(x_i).$$

Similarly $\tau_\rho(F_i) = \dfrac{\alpha_m}{d_{\sigma_m}} \chi_\rho(x_i)$ .

(Note that we already have the relation $\sigma|_K = \rho|_K = \sigma_m|_K$).

Proceeding in the same way, we see that

$$\tau_\sigma(F_{r+1}) = \sum_{j\in J} \delta_j \frac{\alpha_m}{d_{\sigma_m}} \chi_\sigma(s_j) = \frac{\alpha_m}{d_{\sigma_m}} \sum_{j\in J} \delta_j\, \chi_\sigma(s_j)$$

and $$\tau_\rho(F_{r+1}) = \frac{\alpha_m}{d_{\sigma_m}} \sum_{j\in J} \delta_j\, \chi_\rho(s_j)\ .$$

Now for each $i = 1,\ldots,r$

$$\tau_\sigma(F_i) = \tau_\rho(F_i).$$

Therefore $\dfrac{\alpha_m}{d_{\sigma_m}} \chi_\sigma(x_i) = \dfrac{\alpha_m}{d_{\sigma_m}} \chi_\rho(x_i)$ ,

that is, $\chi_\sigma(x_i) = \chi_\rho(x_i) \quad \forall \quad i = 1,\ldots,r.$

We have seen in the proof of Lemma 3.3.1 that if $x \in G$ and $z \in Z$ then there exists a scalar $\lambda_\sigma(z)$ such that $\sigma_{zx} = \lambda_\sigma(z)\ \sigma_x$ and $\lambda_\sigma \in \hat{Z}$. Therefore,

$$\chi_\sigma(zx) = \lambda_\sigma(z)\ \chi_\sigma(x) \text{ and}$$

$$\chi_\sigma(z) = \chi_\sigma(ze) = \lambda_\sigma(z)\ d_\sigma \quad \text{for } z \in Z. \qquad (3.3.1)(c)$$

Since $x_i \in Z$ and $\chi_\sigma(x_i) = \chi_\rho(x_i)$ for $1 \le i \le r$, we get $\lambda_\sigma(x_i) = \lambda_\rho(x_i)$ as $d_\sigma = d_\rho$. Let $h \in H$. Since $H = < x_1,\ldots,x_r,K >$ and each $x_i$ is central,

$$h = x_1^{n_1} \ldots x_r^{n_r} k \text{ for some } k \in K \text{ and } n_1,\ldots,n_r \in \mathbb{Z}.$$

Now, $\chi_\sigma(h) = \chi_\sigma(x_1^{n_1}\ldots x_r^{n_r}k) = \lambda_\sigma(x_1^{n_1}\ldots x_r^{n_r})\ \chi_\sigma(k)$

and $\chi_\rho(h) = \chi_\rho(x_1^{n_1}\ldots x_r^{n_r}k) = \lambda_\rho(x_1^{n_1}\ldots x_r^{n_r})\ \chi_\rho(k)$.

Since $\chi_\sigma(k) = \chi_\rho(k)\ \forall\ k \in K$, $\lambda_\sigma(x_i) = \lambda_\rho(x_i)\ \forall\ i = 1,\ldots,r$ and $\lambda_\sigma,\lambda_\rho \in \hat{Z}$, we see that $\chi_\sigma(h) = \chi_\rho(h)$ for $h \in H$.

Suppose that $\chi_\sigma \ne \chi_\rho$. Since $G = \bigcup_{j\in J} s_j H$, there exists the least index $l$ in J such that $\chi_\sigma$ and $\chi_\rho$ do not agree on $s_l H$. Then

$$\chi_\sigma(s_l h) \ne \chi_\rho(s_l h) \text{ for some } h \in H. \text{ That is,}$$

$$\lambda_\sigma(s_l)\ \chi_\sigma(h) \ne \lambda_\rho(s_l)\ \chi_\rho(h). \quad \text{(Note that each } s_j \text{ is central.)}$$

Since $\chi_\sigma(h) = \chi_\rho(h)$, we have $\lambda_\sigma(s_l) \ne \lambda_\rho(s_l)$ and consequently

$$\chi_\sigma(s_l) \ne \chi_\rho(s_l).$$

Now,

$$|\tau_\sigma(F_{r+1}) - \tau_\rho(F_{r+1})| = \frac{\alpha_m}{d_{\sigma_m}} \left|\sum_{j\ge l} \delta_j\ (\chi_\sigma(s_j) - \chi_\rho(s_j))\right|$$

$$\ge \frac{\alpha_m}{d_{\sigma_m}} \left[\delta_l|\chi_\sigma(s_l) - \chi_\rho(s_l)| - \sum_{j\ge l+1} \delta_j|\chi_\sigma(s_j) - \chi_\rho(s_j)|\right]$$

$$= \alpha_m \left[\delta_l|\lambda_\sigma(s_l) - \lambda_\rho(s_l)| - \sum_{j\ge l+1} \delta_j|\lambda_\sigma(s_j) - \lambda_\rho(s_j)|\right]$$

$$\geq \alpha_m \left[ \delta_l |\overline{\lambda}_\rho \lambda_\sigma(s_l) - 1| - 2 \sum_{j \geq l+1} \delta_j \right] \qquad (3.3.1)(d)$$

$$(\lambda_\rho,\ \lambda_\sigma \in \hat{Z}.)$$

Now $\chi_\sigma = \chi_\rho$ on H. Therefore (3.3.1)(c) implies that

$$\overline{\lambda}_\rho\ \lambda_\sigma = 1 \text{ on } H \cap Z.$$

Next, we define $\gamma : G/H \longrightarrow \mathbb{C}$ by

$$\gamma(s_j H) = \overline{\lambda}_\rho\ \lambda_\sigma(s_j), \qquad j \in J.$$

The definition of $\gamma$ does not depend on particular choice of central representatives of the cosets in G/H. For, if s is another central representative of the coset $s_j H$ then $s_j^{-1}s \in H \cap Z$ and $\overline{\lambda}_\rho\ \lambda_\sigma(s_j) = \overline{\lambda}_\rho\ \lambda_\sigma(s)$. Furthermore, $\gamma$ is multiplicative on G/H because

$$\gamma((s_i H)(s_j H)) = \gamma(s_i s_j H) = \overline{\lambda}_\rho\lambda_\sigma(s_i s_j) = \overline{\lambda}_\rho\lambda_\sigma(s_i)\ \overline{\lambda}_\rho\lambda_\sigma(s_j)$$

$$= \gamma(s_i H)\ \gamma(s_j H). \quad \text{Thus } \gamma \in (G/H)^\wedge.$$

In view of (3.3.1)(d) and the definition of $\gamma$, we have

$$|\tau_\sigma(F_{r+1}) - \tau_\rho(F_{r+1})| \geq \alpha_m \left[ \delta_l |\overline{\lambda}_\rho\lambda_\sigma(s_l) - 1| - 2 \sum_{j \geq l+1} \delta_j \right]$$

$$= \alpha_m \left[ \delta_l |\gamma(s_l H) - 1| - 2 \sum_{j \geq l+1} \delta_j \right]$$

$$\geq \alpha_m \left[ \delta_l \epsilon_l - 2 \sum_{j \geq l+1} \delta_j \right] \qquad \text{(by (3.3.1)(a)}$$

$$> 0. \qquad \text{(by (3.3.1)(b))}.$$

This contradicts the relation $\tau_\sigma(F_{r+1}) = \tau_\rho(F_{r+1})$. Therefore, we must have $\chi_\sigma = \chi_\rho$ on G and $\sigma = \rho$.

This completes the proof.

Remark 3.3.1 Applying Corollary 3.1.5 to the functions constructed in the proof of Theorem 3.3.1, we get a universal Korovkin set in $Z(L^1(G))$ w.r.t. positive operators consisting of $r+4$ functions, where $r$ is the torsion free rank of $G/K$. More precisely, $\left\{ F_o, F_o * F_o, F_o * F_1, \ldots, F_o * F_{r+1}, F_o * \sum_{i=0}^{r+1} F_i * \tilde{F}_i \right\}$ is a universal Korovkin set in $Z(L^1(G))$ w.r.t. positive operators.

Remark 3.3.2 The Gelfand transforms of the functions $F_o, F_1, \ldots, F_{r+1}$, constructed in the proof of Theorem 3.3.1, separate points of $\mathfrak{X}$ strongly. In view of Theorem 2.6.9, $\left\{ F_o, \ldots, F_{r+1}, \sum_{i=0}^{r+1} F_i * \tilde{F}_i \right\}$ is a universal Korovkin set in $Z(L^1(G))$ w.r.t. positive contraction operators.

In Theorem 3.3.1 we have seen that for a [Z] group $G$ having a compact open normal subgroup $K$ such that $G = KZ$, the conditions, (i) $G$ is separable metrizable and (ii) $G/K$ has finite torsion free rank, are sufficient for $Z(L^1(G))$ to have a finite universal Korovkin set. In next theorem we shall show that these conditions are also necessary.

Theorem 3.3.2 Let $G$ be a [Z] group. Suppose $G$ has a compact open normal subgroup $K$ such that $G = KZ$. Then following are equivalent.

(1) $Z(L^1(G))$ admits a finite universal Korovkin set.

(2) $G$ is separable metrizable and $G/K$ has finite torsion free rank.

(3) $\hat{G}$ is finite dimensional separable metric space.

Proof We have proved in Theorem 3.3.1 that (2) implies (1).

Let us now show that (1) implies (3). Suppose $Z(L^1(G))$ admits a finite universal Korovkin set. Then there are finitely many functions in $(Z(L^1(G)))^\wedge$ separating points of the maximal ideal space $\mathcal{X}$. Therefore $\mathcal{X}$ is a finite dimensional separable metric space (Remark 3.1.2). But this is equivalent to saying that $\hat{G}$ is finite dimensional separable metric space.

Finally, we show that (3) implies (2). Suppose $\hat{G}$ is finite dimensional separable metric space. It follows from Theorem 2.3 of [25] that G is second countable. By Theorem 8.3 of [26], G is metrizable.

Next we show that G/K has finite torsion free rank. Since G/K is discrete, $(G/K)^\wedge$ is compact. In view of Theorem 24.28, [26], it suffices to show that $(G/K)^\wedge$ has finite dimension. Now G/K is abelian. Therefore K contains the closure of the commutator subgroup $G'$. Thus $(G/K)^\wedge$ can be identified with a subgroup of $(G/(G')^-)^\wedge$. By Theorem 23.8 of [26], $(G/(G')^-)^\wedge \simeq \left\{\sigma \in \hat{G} : d_\sigma = 1\right\}$ with the relative topology as a subset of $\hat{G}$. Since $\hat{G}$ is finite dimensional, so is $(G/K)^\wedge$.

This completes the proof.

We need the following proposition to show that if G is separable metrizable periodic [Z] group then $Z(L^1(G))$ admits a finite universal Korovkin set.

Proposition 3.3.1 Let G be a periodic [Z] group. Then G contains a compact open normal subgroup K such that G = KZ and G/K is a torsion group.

Proof Since $G$ is a [Z] group, there exists a compact invariant symmetric neighborhood $U$ of $e$ such that $G = UZ$ (see Lemma on p. 365 in [22]). Let $K$ be the subgroup generated by $U$. Then $K$ is an open normal subgroup of $G$ such that $G = KZ$. Moreover, since $K$ is compactly generated and $G$ is periodic, $K$ is compact (see, Corollary 3.14, [24]). Thus $K$ is a compact open normal subgroup of $G$ such that $G = KZ$.

Next, we show that $G/K$ is torsion group. Since $G/K$ is discrete, in view of Theorem 24.26 of [26], it is enough to show that $(G/K)^{\wedge}$ is 0-dimensional. Note that $G/K$ is abelian because $G = KZ$. Therefore $(G/K)^{\wedge}$ can be identified with a subset of $\hat{G}$, as in the proof of Theorem 3.3.2. Now $G$ is periodic, therefore, by Theorem 2.3 of [25], $\hat{G}$ is totally disconnected. Since $\hat{G}$ is also locally compact and Hausdorff, it is 0-dimensional. Therefore $(G/K)^{\wedge}$ is 0-dimensional.

Corollary 3.3.1 Let $G$ be a periodic [Z] group. Then $Z(L^1(G))$ admits a finite universal Korovkin set $\Longleftrightarrow$ $G$ is separable metrizable.

Proof Since $G$ is a periodic [Z] group, by Proposition 3.3.1, there is a compact open normal subgroup $K$ such that $G = KZ$ and torsion free rank of $G/K$ is zero. Corollary now follows from Theorem 3.3.2.

Remark 3.3.3 Let $G$ be a separable metrizable (non compact) periodic [Z] group. Then Proposition 3.3.1 and Remark 3.3.1 imply that $Z(L^1(G))$ has universal Korovkin set w.r.t. positive

operators consisting of four elements.

Furthermore, w.r.t. positive contraction operators $Z(L^1(G))$ has a universal Korovkin set consisting of three elements, see Remark 3.3.2.

Remark 3.3.4 In Proposition 3.3.1 we have shown that if G is a periodic [Z] group then there exists a compact open normal subgroup K such that G/K is a torsion group. We assert that converse of this is also true:

If G contains a compact open normal subgroup K such that G/K is a torsion group then G is periodic.

To prove the assertion, let $x \in G$, we must show that the closed subgroup $[x]$ generated by x is compact. Consider the natural map $\pi : G \longrightarrow G/K$. Then $\pi([x]) \subseteq [\pi(x)]$. Since G/K is discrete and torsion group, $[\pi(x)]$ is finite. Therefore $\pi^{-1}[\pi(x)]$ is compact. Since $[x] \subseteq \pi^{-1}[\pi(x)]$, $[x]$ is compact. This completes the argument.

Now we shall give an example of a [Z] group which is neither compact nor abelian nor a direct product of such groups and to which results of this section apply. Example is due to Grosser and Moskowitz ([24], example 4.7, p. 27).

Example 3.3.1 Let $p \geq 1$ be an integer.

Let $G = \left\{ \begin{bmatrix} 1 & b & a \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : a \in \Omega_p,\ b,c \in \Delta_p \right\}$, where $\Omega_p$ is

the locally compact ring of p-adic numbers and $\Delta_p$ is the ring of p-adic integers which is, in fact, a compact open subring of $\Omega_p$, see, Section 10 of [26]. It is easy to see that

$$Z(G) = \left\{ \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : a \in \Omega_p \right\}, \text{ so that } G/Z(G) \simeq \Delta_p \times \Delta_p$$

which is compact. Thus G is a [Z] group but is neither compact nor abelian nor a direct product of such groups. Moreover, G is separable metrizable since $\Omega_p$ and $\Delta_p$ are so.

Let $K = \left\{ \begin{bmatrix} 1 & b & a \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in \Delta_p \right\}$. It is clear that K is a compact open normal subgroup of G. Furthermore, G = KZ and G/K is an abelian torsion group, i.e., torsion free rank of G/K is zero. Applying Theorem 3.3.1, $Z(L^1(G))$ admits a finite universal Korovkin set; in fact, a four element set w.r.t. positive operators and a three element set w.r.t. positive contraction operators, see Remark 3.3.1 and Remark 3.3.2.

Note that G is a periodic [Z] group, we could, therefore, apply Corollary 3.3.1 as well.

# CHAPTER IV

## KOROVKIN THEORY FOR POSITIVE OPERATORS ON SEGAL ALGEBRAS

Throughout this chapter, unless otherwise specified, $S(G)$ will denote a Segal algebra over a non discrete locally compact group $G$, which is closed under involution ~. (Recall that $f^{\sim}(x) = f(x^{-1})$). Note that there exist Segal algebras which are not closed under the involution ~, see for example, Johnson [27]. For definition and some examples of Segal algebras we refer §2.5 of Chapter II. It is well known that if $G$ is abelian then $S(G)$ is a commutative Banach algebra with continuous symmetric involution. If $G$ is non abelian then $Z(S(G))$, the centre of $S(G)$, is nontrivial iff $G$ is an [IN] group (See, [48], Corollary 3, p. 64). In this case $Z(S(G))$ is a commutative Banach algebra with continuous symmetric involution.

Results proved by Pannenberg in [45], for the algebra $L^1(G)$ ($G$ locally compact abelian group) and the results of preceeding chapter motivate us to ask the following natural questions:

Question 4.1.1 If $G$ is a locally compact abelian group, when does $S(G)$ admit a finite universal Korovkin set w.r.t. positive operators?

Question 4.1.2 If $G$ is a non abelian [IN] group, when does $Z(S(G))$ admit a finite universal Korovkin set w.r.t. positive operators?

In this chapter, we attempt to investigate these questions and reveal certain distinctive features of Segal algebras in the framework of Korovkin theory.

In the preceeding chapter we studied Banach algebras satisfying property 'P'. For a commutative Banach algebra A with continuous symmetric involution and satisfying property 'P', we saw in Corollary 3.1.5 that if S is a subset of A such that $\hat{S}$ separates points of $\Delta A$ and if x is an element in A such that $\hat{x} > 0$ then $\{x\} \cup xS \cup xSS^*$ is a universal Korovkin set in A w.r.t. positive operators. In Section 1, we show that a Segal algebra need not satisfy property 'P' and therefore Corollary 3.1.5 can not be applied to yield a universal Korovkin set in a Segal algebra. However, we prove by different techniques (in Section 2) that a Segal algebra S(G) over a compact metrizable group G has a finite universal Korovkin set and thereby give a complete answer to the Question 4.1.1 for the case when G is a compact abelian group. For a compact group G, a complete answer to Question 4.1.2 is provided in Section 2.

In ([2], Corollary 4.3) Altomare has proved that if A is a commutative Banach algebra with continuous symmetric involution and a bounded approximate identity uniformly bounded by one and if S is a subset of A such that $\hat{S}$ separates points of $\Delta A$ strongly then $S \cup SS^*$ is a universal Korovkin set w.r.t. positive contraction operators. In Section 1, we extend this result to the commutative Banach algebras with continuous symmetric involution and satisfying a certain property '$P_1$' (to be specified

in Section 1). We also show by an example that in the absence of property '$P_1$', the result need not be true.

## §1

As we shall have occasions to deal with continuous positive functionals of a Segal algebra, it is necessary for us to know a suitable description of them. We begin with such a description for a Segal algebra S(G) over a compact abelian group G. It is clear that S(G) contains all the characters of G. Let $\phi$ be a continuous functional on S(G). Fourier coefficients of $\phi$ are defined by

$$\hat{\phi}(\gamma) = \langle\gamma,\phi\rangle \ , \ \gamma \in \hat{G} \qquad \text{(see, [28], §7.1, p. 34).}$$

We have the following simple Lemma.

Lemma 4.1.1 Let S(G) be a Segal algebra over a compact abelian group G and let $\phi$ be a continuous functional on S(G). Then $\phi$ is positive iff $\hat{\phi}(\gamma) \geq 0 \ \forall \ \gamma \in \hat{G}$.

Proof By definition, $\phi$ is positive iff $\langle f * \tilde{f}, \phi\rangle \geq 0 \ \forall \ f \in S(G)$. Since $\mathcal{T}(G)$, the set of all trigonometric polynomials, is dense in S(G), this is equivalent to $\langle f * \tilde{f},\phi\rangle \geq 0 \ \forall \ f \in \mathcal{T}(G)$

or equivalently, $\qquad \langle\gamma * \tilde{\gamma},\phi\rangle \geq 0 \quad \forall \ \gamma \in \hat{G}$.

That is, $\qquad \langle\gamma,\phi\rangle \geq 0 \quad \forall \ \gamma \in \hat{G}$.

Now we provide some examples of Segal algebras which fail to satisfy property 'P'.

Example 4.1.1 Let $G$ be a compact abelian group and $S(G)$ be any of the following Segal algebras:

(i) $L^p(G)$, $(1 < p < \infty)$, with norm $\| \ \|_p$.

(ii) For $1 \le p < \infty$ and $\mu$ an unbounded positive regular Borel measure on $\hat{G}$, $A^p_\mu(G) = \left\{f \in L^1(G) : \hat{f} \in L^p(\hat{G},\mu)\right\}$ with norm $\|f\|_{A^p_\mu} = \|f\|_1 + \|\hat{f}\|_{L^p(\hat{G},\mu)}$.

Then $S(G)$ does not satisfy property 'P'.

Proof Suppose that $S(G)$ satisfies property 'P'. Let $\phi$ be a continuous positive functional on $S(G)$. Then there exists a constant $C$ such that

$$|\langle f,\phi\rangle| \le C\|\hat{f}\|_\infty \text{ for all } f \in S(G).$$

Let $\gamma_1,\ldots,\gamma_n \in \hat{G}$ and $f = \sum_{i=1}^{n} a_i\gamma_i$, where $a_i = \operatorname{sgn} \langle\gamma_i,\phi\rangle$. Then we have $\|\hat{f}\|_\infty = 1$ and $\langle f,\phi\rangle = \sum_{i=1}^{n} \operatorname{sgn}\langle\gamma_i,\phi\rangle \langle \gamma_i,\phi\rangle = \sum_{i=1}^{n} |\langle\gamma_i,\phi\rangle|$. Therefore $\sum_{i=1}^{n} |\langle\gamma_i,\phi\rangle| = |\langle f,\phi\rangle| \le C\|\hat{f}\|_\infty = C$.

This is true for all $n \in \mathbb{N}$. Therefore $\sum_{\gamma\in\hat{G}} |\langle\gamma,\phi\rangle| \le C$. Thus if $S(G)$ satisfies property 'P' then for every continuous positive functional $\phi$ on $S(G)$

$$\sum_{\gamma\in\hat{G}} |\langle\gamma,\phi\rangle| < \infty \qquad (4.1.1)(a)$$

We shall now show that if $S(G)$ is either of the algebras (i) or (ii), there is a continuous positive functional $\phi$ on $S(G)$ which does not satisfy the condition (4.1.1)(a).

First, we consider the case of $L^p(G)$, $1 < p \le 2$.

Let $\{\gamma_n\}_{n=1}^{\infty}$ be a sequence of distinct characters in $\hat{G}$. Let h be the function on $\hat{G}$ defined by

$$h(\gamma) = \begin{cases} \frac{1}{n} & \text{if } \gamma = \gamma_n \\ 0 & \text{if } \gamma \notin \bigcup_{n=1}^{\infty} \{\gamma_n\} . \end{cases}$$

It is clear that $h \notin L^1(\hat{G})$ and $h \in L^p(\hat{G})$ $\forall$ $p > 1$. Since $1 < p \le 2$ and G is compact, there exists $g \in L^q(G)$, where $\frac{1}{p} + \frac{1}{q} = 1$, such that $\hat{g} = h$. Since $\hat{g} \ge 0$, g defines a continuous positive functional on $L^p(G)$, (Lemma 4.1.1). Moreover, $\sum_{\gamma\in\hat{G}} |\langle\gamma,g\rangle| = \sum_{n=1}^{\infty} \frac{1}{n}$ $= \infty$ so that the condition (4.1.1)(a) is not satisfied.

Now we consider the case of $L^p(G)$, $p > 2$. Let q be the conjugate exponent of p. Since $p > 2$ we have $q < 2$. By above arguments there exists $g \in L^p(G)$ such that g defines a continuous positive functional on $L^q(G)$ which does not satisfy condition (4.1.1)(a). But since $L^p(G) \subseteq L^q(G)$, g also defines a continuous positive functional on $L^p(G)$.

Next, we deal with the case of $A^p_\mu(G)$, $1 \le p < \infty$.

Let us first see how a continuous positive functional on $A^p_\mu(G)$ looks like. Since the map $f \longrightarrow (f,\hat{f})$ is a linear isometry of $A^p_\mu(G)$ into $L^1(G) \times L^p(\hat{G},\mu)$, $A^p_\mu(G)$ may be regarded as a closed linear subspace of $L^1(G) \times L^p(\hat{G},\mu)$. An application of Hahn Banach Theorem shows that every continuous functional $\phi$ on $A^p_\mu(G)$ arises from a pair $(g,h) \in L^{\infty}(G) \times L^q(\hat{G},\mu)$ such that

$$<f,\phi> = <f,g> + <\hat{f},h> \qquad \forall\ f \in A^p_\mu(G).$$

Let $\phi$ be a continuous functional on $A^p_\mu(G)$ arising from the pair $(g,h) \in L^\infty(G) \times L^q(\hat{G},\mu)$. Then for $\gamma \in \hat{G}$

$$<\gamma,\phi> = <\gamma,g> + <\hat{\gamma},h>$$

$$= <\gamma,g> + \int_{\hat{G}} \hat{\gamma}(\eta)\ h(\eta)\ d\mu(\eta)$$

$$= <\gamma,g> + h(\gamma)\ \mu(\{\gamma\}).$$ Thus, in view of Lemma 4.1.1, $\phi$ is a positive functional $\iff \hat{g}(\gamma) + h(\gamma)\ \mu(\{\gamma\}) \geq 0 \quad \forall\ \gamma \in \hat{G}$.

Let $\left\{F_n\right\}_{n=1}^\infty$ be a sequence of disjoint subsets of $\hat{G}$ such that

$\mu(F_n) \geq n \quad \forall\ n \in \mathbb{N}$. Let us write $a_n$ for $\mu(F_n)$.

We define h on $\hat{G}$ by

$$h(\gamma) = \begin{cases} \dfrac{1}{n\ a_n} & \text{if } \gamma \in F_n \\ 0 & \text{if } \gamma \notin \bigcup_n F_n \end{cases}.$$

Then $h \in L^q(\hat{G},\mu) \quad \forall\ q > 1$. In fact,

$$\|h\|^q_{L^q(\hat{G},\mu)} = \sum_{\gamma\in\hat{G}} |h(\gamma)|^q\ \mu(\{\gamma\}) = \sum_n \sum_{\gamma\in F_n} \frac{1}{n^q\ a_n^q}\ \mu(\{\gamma\})$$

$$= \sum_n \frac{1}{n^q\ a_n^q}\ a_n = \sum_n \frac{1}{n^q\ a_n^{q-1}} \leq \sum_n \frac{1}{n^q} < \infty.$$

But $h \notin L^1(\hat{G},\mu)$ because $\|h\|_{L^1(\hat{G},\mu)} = \sum_{\gamma\in\hat{G}} |h(\gamma)|\ \mu(\{\gamma\})$

$$= \sum_n \sum_{\gamma\in F_n} \frac{1}{n\ a_n}\ \mu(\{\gamma\}) = \sum_n \frac{1}{n\ a_n}\ a_n = \sum_n \frac{1}{n} = \infty.$$

The functional defined by the pair $(0,h) \in L^\infty(G) \times L^q(\hat{G},\mu)$ is clearly a continuous positive functional on $A^p_\mu(G)$ not satisfying the condition (4.1.1)(a).

Let us say that a commutative Banach algebra A with a continuous involution satisfies property '$P_1$' if its every continuous positive functional $\phi$ satisfies

$$|\phi(x)| \le \|\phi\| \; \|\hat{x}\|_\infty \quad \forall \; x \in A.$$

It is known that every commutative Banach algebra with continuous symmetric involution and having an approximate identity $\{e_\alpha\}$ such that $\operatorname{Sup}_\alpha \|e_\alpha\| \le 1$ satisfies property '$P_1$', see [37], p. 275. In fact, it can be seen that a commutative Banach algebra with continuous symmetric involution for which every continuous positive functional is extendable without increasing its norm satisfies property '$P_1$'.

Following the arguments used in Corollary 3.1.2, we can prove the following corollary of Theorem 2.6.6 (Altomare, [2], Theorem 4.1).

Corollary 4.1.1 Let A be a commutative Banach algebra with continuous symmetric involution and satisfying property '$P_1$'. Let S be a subset of A. Then S is a universal Korovkin set in A w.r.t. positive contraction operators iff $\hat{S}$ is a universal Korovkin set in $C_o(\Delta A)$ w.r.t. positive contraction operators.

Theorem 2.6.8 and Corollary 4.1.1 now give the following analogue of Theorem 2.6.9.

Corollary 4.1.2 Let A be a commutative Banach algebra with continuous symmetric involution and satisfying property '$P_1$'. Let S be a subset of A such that $\hat{S}$ separates points of $\Delta A$ strongly.

Then $S \cup SS^*$ is a universal Korovkin set in A w.r.t. positive contraction operators.

The following example shows that Corollary 4.1.2 may not be true for algebras not satisfying property '$P_1$'. We have already seen in Example 4.1.1 that $L^p(G)$, $1 < p < \infty$, does not satisfy property '$P_1$'.

Example 4.1.2 Let G be a compact abelian metrizable group. Consider $L^p(G)$, $1 < p < \infty$. Let $\left\{\gamma_n\right\}_{n=1}^{\infty}$ be an enumeration of $\hat{G}$ and $h = \sum_{n=1}^{\infty} \frac{1}{2^n} \gamma_n$. Then $h \in L^p(G)$ and $\hat{h}$ separates points of $\hat{G}$ strongly. We shall prove that the set $\left\{ h, h * \tilde{h} \right\}$ is not a universal Korovkin set in $L^p(G)$ w.r.t. positive contraction operators.

In view of Theorem 2.6.6, $\left\{ h, h * \tilde{h} \right\}$ will be a universal Korovkin set in $L^p(G)$ w.r.t. positive contraction operators iff for every $\gamma \in \hat{G} \cup \{0\}$ and for every continuous positive functional $\phi$ on $L^p(G)$ such that $\|\phi\| \le 1$, $\langle h,\phi\rangle = \hat{h}(\gamma)$ and $\langle h * \tilde{h},\phi\rangle = |\hat{h}(\gamma)|^2$ imply that $\phi = \gamma$. (Note that $\hat{h}(0) = 0$.)

First we consider the case of $L^p(G)$, $1 < p \le \frac{3}{2}$.

Let q be the conjugate exponent of p. Since $p \le \frac{3}{2}$ we have $q \ge 3$.

Let $g = \frac{1}{8^{q-1}} \gamma_1 + \left(1 - \frac{6}{8^{q-1}}\right) \gamma_2 + \frac{1}{8^{q-2}} \gamma_3$. Then g belongs to $L^q(G)$ and $\hat{g} \ge 0$. Therefore g defines a continuous positive functional on $L^p(G)$ (Lemma 4.1.1).

We now show that $\|g\|_q \le 1$. By Hausdorff Young Theorem,

$$\|g\|_q \le \|\hat{g}\|_p.$$

Now,

$$\|\hat{g}\|_p^p = \|\hat{g}\|_{q/q-1}^{q/q-1} = \sum_{\eta\in\hat{G}} |\hat{g}(\eta)|^{q/q-1}$$

$$= \left(\frac{1}{8^{q-1}}\right)^{q/q-1} + \left(1 - \frac{6}{8^{q-1}}\right)^{q/q-1} + \left(\frac{1}{8^{q-2}}\right)^{q/q-1}$$

$$= \frac{1}{8^q} + \left(1 - \frac{6}{8^{q-1}}\right)^{q/q-1} + \frac{8^{\frac{1}{q-1}}}{8^{q-1}}$$

$$\left(\text{note that } \frac{(q-2)q}{q-1} = (q-1) - \frac{1}{(q-1)}\right)$$

$$\leq \frac{1}{8^{q-1}} + \left(1 - \frac{6}{8^{q-1}}\right) + \frac{8^{1/2}}{8^{q-1}} \qquad (q \geq 3)$$

$$\leq \frac{1}{8^{q-1}} + 1 - \frac{6}{8^{q-1}} + \frac{3}{8^{q-1}}$$

$$= 1 - \frac{2}{8^{q-1}} < 1.$$

Therefore $\|g\|_q \leq \|\hat{g}\|_p \leq 1$. Thus $g$ defines a continuous positive functional on $L^p(G)$ which has norm at most one. Moreover, it can be seen that $g$ satisfies

$$\langle h,g\rangle = \frac{1}{4} = \hat{h}(\gamma_2)$$

and $$\langle h * \tilde{h},g\rangle = \frac{1}{16} = |\hat{h}(\gamma_2)|^2$$

but $g \neq \gamma_2$. Therefore, $\left\{ h, h * \tilde{h} \right\}$ is not a universal Korovkin set w.r.t. positive contraction operators in $L^p(G)$, $1 < p \leq \frac{3}{2}$. Now we consider the case when $p > \frac{3}{2}$. Clearly we have $q < 3$. We choose a number $k \geq 3$. Let

$$g = \frac{1}{8^{k-1}}\gamma_1 + \left(1 - \frac{6}{8^{k-1}}\right)\gamma_2 + \frac{1}{8^{k-2}}\gamma_3.$$ By the above

computations we have $\|g\|_k \le 1$. Since $q \le k$, we have $\|g\|_q \le \|g\|_k \le 1$. Thus g defines a continuous positive functional on $L^p(G)$ with norm at most one and satisfies $\langle h,g\rangle = \hat{h}(\gamma_2)$ and $\langle h * \hat{h},g\rangle = |\hat{h}(\gamma_2)|^2$. Since $g \ne \gamma_2$, the desired conclusion follows.

Remark 4.1.1 Let G be compact abelian metrizable group and h be the function as in Example 4.1.2. Similar arguments may be applied to show that $\left\{ h, h * \tilde{h} \right\}$ is not a universal Korovkin set w.r.t. positive contraction operators in the Segal algebras C(G) and $A_p(G)$, $1 \le p < \infty$.

Furthermore, since $\left\{\hat{h},\ |\hat{h}|^2\right\}$ is a universal Korovkin set in $C_o(\hat{G})$ w.r.t. positive contraction operators, Example 4.1.2 also shows that Corollary 4.1.1 need not hold if A does not satisfy property '$P_1$'.

Remark 4.1.2 Let G be compact abelian metrizable group and h be the function as in Example 4.1.2. Let us compare behaviour of $L^1(G)$ and certain Segal algebras over G. The set $\left\{ h,\ h * \tilde{h} \right\}$ is a universal Korovkin set w.r.t. positive contraction operators in $L^1(G)$, while this is not the case with $L^p(G)$ $(1 < p < \infty)$, C(G) or $A_p(G)$ $(1 \le p < \infty)$. What is the story w.r.t. positive operators? We know that the set $\left\{h,\ h * h,\ h * h * \tilde{h}\right\}$ is a universal Korovkin set in $L^1(G)$ w.r.t. positive operators (Example 3.1.3). In view of this, one may ask whether the set $\left\{h,\ h * h,\ h * h * \tilde{h}\right\}$ is a universal Korovkin set w.r.t. positive operators in $L^p(G)$ $(1<p<\infty)$, C(G) or $A_p(G)$ $(1 \le p < \infty)$. In view of Example 4.1.1, the

Corollary 3.1.5 is no longer at our disposal. Nevertheless, we shall see that $\{h, h * h, h * h * \tilde{h}\}$ is a universal Korovkin set in $L^p(G)$ $(1 < p < \infty)$, $C(G)$ or $A_p(G$ $(1 \leq p < \infty)$. In fact, it is a universal Korovkin set in any Segal algebra which contains this set, (see, Proof of Theorem 4.2.1).

§2.

In this section we attempt to investigate Question 4.1.1 and Question 4.1.2. We begin with an answer of Question 4.1.1 for the case when G is a compact abelian group. We prove the following theorem.

Theorem 4.2.1 Let S(G) be a Segal algebra over a compact abelian group G. Then S(G) admits a finite (three element) universal Korovkin set w.r.t. positive operators iff G is metrizable.

Proof Suppose that S(G) has a finite universal Korovkin set. Then finitely many functions in $(S(G))\hat{}$ separate points of $\hat{G}$. Therefore $\hat{G}$ is second countable (Remark 3.1.2). Since $\hat{G}$ is discrete, it is countable. It follows that G is metrizable.

Conversely, suppose G is metrizable. Then $\hat{G}$ is countable. Let $\{\gamma_n\}_{n=1}^{\infty}$ be an enumeration of $\hat{G}$. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of distinct positive numbers such that $\sum\limits_{n=1}^{\infty} a_n \|\gamma_n\|_S < \infty$. It is clear that the function $h = \sum\limits_{n=1}^{\infty} a_n\gamma_n$ belongs to S(G). We shall prove that the set $\{h, h * h, h * h * h\} = \left\{\sum\limits_{n=1}^{\infty} a_n\gamma_n, \sum\limits_{n=1}^{\infty} a_n^2\gamma_n, \sum\limits_{n=1}^{\infty} a_n^3\gamma_n\right\}$

is a universal Korovkin set in $S(G)$ w.r.t. positive operators.

Let $m \in \mathbb{N}$ and $\phi$ be a continuous positive functional on $S(G)$ which satisfy following three equations

$$\langle h,\phi\rangle = \langle h,\gamma_m\rangle$$

$$\langle h * h, \phi\rangle = \langle h * h, \gamma_m\rangle$$

and $$\langle h * h * h,\phi\rangle = \langle h * h * h, \gamma_m\rangle$$

or equivalently,

$$\sum_{n\in\mathbb{N}} a_n \langle\gamma_n,\phi\rangle = a_m$$

$$\sum_{n\in\mathbb{N}} a_n^2 \langle\gamma_n,\phi\rangle = a_m^2$$

and $$\sum_{n\in\mathbb{N}} a_n^3 \langle\gamma_n,\phi\rangle = a_m^3.$$

In view of Theorem 2.6.5, we must show that $\phi = \gamma_m$. Let us write $b_n$ for $\langle\gamma_n,\phi\rangle$. Since $\phi$ is a continuous positive functional on $S(G)$, $b_n \geq 0 \quad \forall\ n \in \mathbb{N}$ (Lemma 4.1.1). The three equations now can be written as

$$\sum_{n\in\mathbb{N}-\{m\}} a_n b_n = a_m(1 - b_m)$$

$$\sum_{n\in\mathbb{N}-\{m\}} a_n^2 b_n = a_m^2(1 - b_m) \qquad (4.2.1)(a)$$

and $$\sum_{n\in\mathbb{N}-\{m\}} a_n^3 b_n = a_m^3(1 - b_m) .$$

These equations imply that

$$a_m^2(1 - b_m) = \sum_{n\in\mathbb{N}-\{m\}} a_n^2 b_n = \sum_{n\in\mathbb{N}-\{m\}} (a_n^{3/2}b_n^{1/2})(a_n^{1/2}b_n^{1/2})$$

$$\leq \left(\sum_{n\in\mathbb{N}-\{m\}} a_n^3 b_n\right)^{1/2} \left(\sum_{n\in\mathbb{N}-\{m\}} a_n b_n\right)^{1/2}$$

$$= \left(a_m^3(1 - b_m)\right)^{1/2} \left(a_m(1 - b_m)\right)^{1/2}$$

$$= a_m^2(1 - b_m).$$

Therefore, the equality holds throughout. Thus there exists a nonnegative constant C such that

$$a_n^3 b_n = C\, a_n b_n \qquad \forall\ n \in \mathbb{N} - \{m\}. \tag{4.2.1)(b}$$

We shall prove that $b_n = 0 \ \forall\ n \in \mathbb{N} - \{m\}$. Suppose that $b_k \neq 0$ for some $k \in \mathbb{N} - \{m\}$. Then the equality $a_k^3 b_k = C\, a_k b_k$ implies that $a_k^2 = C$. Since $a_n$'s are distinct positive numbers, it follows that $b_n = 0 \ \forall\ n \in \mathbb{N} - \{m,k\}$. First two equations in (4.2.1)(a) now reduce to

$$a_k b_k = a_m(1 - b_m)$$

and $a_k^2 b_k = a_m^2(1 - b_m)$. This implies that

$$a_k a_m(1 - b_m) = a_m^2(1 - b_m).$$

Hence $a_k(1 - b_m) = a_m(1 - b_m)$. Since $a_k \neq a_m$, we must have $b_m = 1$ and therefore $b_k = 0$. Thus $b_n = 0$ for all $n \neq m$. This implies that $\phi = \gamma_m$.

This completes the proof.

Let G be a compact group. In [48], (Corollary 6, p. 65) it is shown that the maximal ideal space of Z(S(G)) can be identified with $\mathcal{X} = \left\{\frac{\chi_\sigma}{d_\sigma} : \sigma \in \hat{G}\right\}$ with discrete topology. For $\sigma \in \hat{G}$, $\tau_\sigma$ will denote the multiplicative functional on Z(S(G)) defined by

$$\langle f,\tau_\sigma\rangle = \int_G f(x)\frac{\overline{\chi_\sigma(x)}}{d_\sigma}\,dx, \qquad f \in Z(S(G)).$$

Let $Z(\mathcal{T}(G))$ denotes the linear span of the set $\{\chi_\sigma : \sigma \in \hat{G}\}$. It can be seen that $Z(S(G))$ contains $Z(\mathcal{T}(G))$ and $Z(\mathcal{T}(G))$ is dense in $Z(S(G))$.

Similar arguments as in Lemma 4.1.1 can be applied to prove the following

Lemma 4.2.1 Let G be a compact group. A continuous functional $\phi$ on $Z(S(G))$ is positive iff $\langle\chi_\sigma,\phi\rangle \geq 0 \;\; \forall\, \sigma \in \hat{G}$.

We have the following answer to Question 4.1.2 for the case when G is a compact group.

Theorem 4.2.2 Let G be a compact group. Then $Z(S(G))$ admits a finite (three element) universal Korovkin set w.r.t. positive operators iff G is metrizable.

Proof Necessity part can be proved as in Proposition 3.2.1. For the sufficiency, we suppose G is metrizable. Then $\hat{G}$ is countable. Let $\{\sigma_n\}_{n=1}^{\infty}$ be an enumeration of $\hat{G}$. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of distinct positive numbers such that $\sum_{n=1}^{\infty} a_n d_{\sigma_n} \|\chi_{\sigma_n}\|_S < \infty$, where $\|\;\;\|_S$ denotes the norm in $S(G)$. It is clear that the function $h = \sum_{n=1}^{\infty} a_n d_{\sigma_n} \chi_{\sigma_n}$ belongs to $Z(S(G))$. We shall prove that the set $\{h,\ h*h,\ h*h*h\} = \left\{\sum_{n=1}^{\infty} a_n d_{\sigma_n}\chi_{\sigma_n},\ \sum_{n=1}^{\infty} a_n^2 d_{\sigma_n}\chi_{\sigma_n},\ \sum_{n=1}^{\infty} a_n^3 d_{\sigma_n}\chi_{\sigma_n}\right\}$ is a universal Korovkin set in $Z(S(G))$ w.r.t. positive operators.

Let $m \in \mathbb{N}$ of $\phi$ be a continuous positive functional on $Z(S(G))$ such that

$$<h,\phi> = <h,\tau_{\sigma_m}>$$

$$<h * h,\phi> = <h * h, \tau_{\sigma_m}>$$

and $$<h * h * h,\phi> = <h * h * h, \tau_{\sigma_m}> .$$

In view of Theorem 2.6.5, it suffices to show that $\phi = \tau_{\sigma_m}$. In view of orthogonality relation of characters, above three equations reduce to

$$\sum_{n=1}^{\infty} a_n d_{\sigma_n} <\chi_{\sigma_n},\phi> = a_m$$

$$\sum_{n=1}^{\infty} a_n^2 d_{\sigma_n} <\chi_{\sigma_n},\phi> = a_m^2 \qquad (4.2.2)(a)$$

$$\sum_{n=1}^{\infty} a_n^3 d_{\sigma_n} <\chi_{\sigma_n},\phi> = a_m^3 .$$

Let us write $b_n$ for $d_{\sigma_n} < \chi_{\sigma_n},\phi>$. Since $\phi$ is a continuous positive functional on $Z(S(G))$, by Lemma 4.2.1, $b_n \geq 0 \ \forall \ n \in \mathbb{N}$. (4.2.2)(a) now can be written as

$$\sum_{n\in\mathbb{N}-\{m\}} a_n b_n = a_m(1 - b_m)$$

$$\sum_{n\in\mathbb{N}-\{m\}} a_n^2 b_n = a_m^2(1 - b_m) \qquad (4.2.2)(b)$$

$$\sum_{n\in\mathbb{N}-\{m\}} a_n^3 b_n = a_m^3(1 - b_m) .$$

Note that equations in (4.2.2)(b) are same as those in (4.2.1)(a) (Proof of Theorem 4.2.1). Now proceeding exactly as in the proof

of Theorem 4.2.1, we get

$$b_n = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{otherwise.} \end{cases}$$

That is, $d_{\sigma_n} \langle \chi_{\sigma_n}, \phi \rangle = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{otherwise.} \end{cases}$

Therefore, it follows that $\phi = \tau_{\sigma_m}$.

This completes the proof.

For the case when G is a non compact locally compact abelian group, it seems difficult to tackle Question 4.1.1. Main obstacle is that we do not know a handy description of continuous positive functionals of $S(G)$, even for the Segal algebras $C_o(G) \cap L^1(G)$, $L^p(G) \cap L^1(G)$ $(1 < p < \infty)$ or $A_p(G)$ $(1 \le p < \infty)$. However, we remark the following.

Remark 4.2.1 Let G be a non compact locally compact abelian group. Consider the Segal algebra $C_o(G) \cap L^1(G)$. We may consider $C_o(G) \cap L^1(G)$ with another equivalent norm given by $\max\left\{\| \ \|_\infty, \| \ \|_1\right\}$.

In view of Theorem 1.7 of [34], dual of $C_o(G) \cap L^1(G)$ is isometrically isomorphic to $(M(G) \times L^\infty(G))/K$, where

$$K = \left\{(\mu,\phi) \in M(G) \times L^\infty(G) : \langle f,\mu\rangle = \langle f,\phi\rangle \ \forall\, f \in C_o(G) \cap L^1(G)\right\}$$

and norm in $(M(G) \times L^\infty(G))/K$ is given by

$$\|\Phi\| = \inf\left\{\|\mu\| + \|\phi\| : (\mu,\phi) \in M(G) \times L^\infty(G),\ \Phi \equiv (\mu,\phi) \bmod K\right\}.$$

Let $\Phi$ be a continuous functional on $C_o(G) \cap L^1(G)$. If $\Phi \equiv (\mu,\phi)$ mod K for some $(\mu,\phi) \in M(G) \times L^\infty(G)$ such that $\hat{\mu} \geq 0$ and $\phi$ is a continuous positive definite function on G then clearly $\Phi$ is a continuous positive functional on $C_o(G) \cap L^1(G)$. If every continuous positive functional $\Phi$ on $C_o(G) \cap L^1(G)$ is of the form $(\mu,\phi)$ mod K with $\hat{\mu} \geq 0$ and $\phi$ continuous positive definite then it can be shown that

$C_o(G) \cap L^1(G)$ admits a finite universal Korovkin set w.r.t. positive operators iff $\hat{G}$ is finite dimensional separable metric space.

# CHAPTER V

# KOROVKIN THEORY FOR POSITIVE SPECTRAL CONTRACTION OPERATORS

In this chapter we study universal Korovkin sets w.r.t. positive spectral contraction operators in a commutative Banach algebra A with continuous symmetric involution. Universal Korovkin sets w.r.t. positive spectral contraction operators in the algebras admitting an identity or a bounded approximate identity have been studied by many authors, for example, see Altomare [1], Pannenberg [46], Pannenberg and Romanelli [47], Romanelli [55]. We shall see that if A satisfies property '$P_1$' then universal Korovkin sets w.r.t. positive spectral contraction operators coincide with universal Korovkin sets w.r.t. positive contraction operators and in the absence of property '$P_1$' the two notions are different. We prove that in a commutative Banach algebra A with continuous symmetric involution a subset S is a universal Korovkin set w.r.t. positive spectral contraction operators iff $\hat{S}$ is a universal Korovkin set in $C_o(\Delta A)$ w.r.t. positive contraction operators. As a consequence we obtain a characterization of locally compact abelian groups G for which a Segal algebra (closed under involution) over G admits a finite universal Korovkin set w.r.t. positive spectral contraction operators.

Throughout this chapter, A denotes a commutative Banach algebra with continuous symmetric involution. For a subset S of A, let $\hat{\text{Kor}}_u^{1+}(S)$ denote the universal Korovkin closure of S in A w.r.t. positive spectral contraction operators. This, by definition, is the set of all $y \in A$ such that if B is a commutative Banach algebra with continuous symmetric involution, T is an algebraic *-homomorphism from A to B and if a net $\{T_\alpha\}$ of positive spectral contraction operators from A to B satisfy $\lim_\alpha \|(T_\alpha x - Tx)^\wedge\|_\infty = 0 \ \forall\ x \in S$ then $\lim_\alpha \|(T_\alpha y - Ty)^\wedge\|_\infty = 0$, see §2.6.

Let $\Phi_1^+(A)$ denote the set $\left\{\phi \in A' : \phi \text{ is positive and } |\phi(x)| \le \|\hat{x}\|_\infty \ \forall\ x \in A\right\}$. It can be seen easily that $\Phi_1^+(A)$ is a $\omega^*$-compact subset of $A'$. Following proposition can be deduced from Corollary 1.2 of Altomare [2]. However, we give a proof which is based on techniques similar to those in the proof of Theorem 1.1(2) of Romanelli [55]. We observe that the condition of existence of bounded approximate identity in the algebra A is redundant in Theorem 1.1(2) of [55]; see Proposition 5.1.2 and also Remark 5.1.1.

Proposition 5.1.1 Let A be a commutative Banach algebra with continuous symmetric involution and S be a subset of A. Then

$$\hat{\text{Kor}}_u^{1+}(S) = \Big\{y \in A : \text{if } m \in \Delta_\omega A \text{ and } \phi \in \Phi_1^+(A) \text{ satisfy } \phi(x) = m(x) \ \forall\ x \in S \text{ then } \phi(y) = m(y)\Big\}.$$

Proof Let us denote the set on right side by U. The inclusion $\hat{\mathrm{Kor}}_u^{1+}(S) \subseteq U$ follows immediately from the definition of $\hat{\mathrm{Kor}}_u^{1+}(S)$; one only needs to check the definition for $B = \mathbb{C}$, the algebraic $*$-homomorphism $m : A \longrightarrow \mathbb{C}$ and the constant net $\{\phi\}$ of positive spectral contraction operators from A to $\mathbb{C}$.

To prove the reverse inclusion, let $y \in U$. Let B be a commutative Banach algebra with continuous symmetric involution, $T: A \longrightarrow B$ be an algebraic $*$-homomorphism and $\{T_i\}_{i\in I}$ be a net of positive spectral contraction operators from A to B satisfying $\lim_{i\in I} \|(T_i x - Tx)^\wedge\|_\infty = 0 \quad \forall\, x \in S$. Suppose that $\lim_{i\in I} \|(T_i y - Ty)^\wedge\|_\infty \neq 0$. Then there exists an $\epsilon_o > 0$ and for each $i \in I$ there exists $\alpha_i \in I$ such that $\alpha_i \geq i$ and $\eta_i \in \Delta B$ such that

$$|\eta_i(T_{\alpha_i} y) - \eta_i(Ty)| > \epsilon_o \tag{5.1.1}$$

Now $\{\eta_i\}_{i\in I}$ is a net in $\Delta_\omega B$. Since $\Delta_\omega B$ is $\omega^*$-compact, there exists a $\eta \in \Delta_\omega B$ and a subnet $\{\eta_i\}_{i\in I_o}$ which converges to $\eta$ in $\omega^*$-topology of $B'$. For each $i \in I_o$, let us write $\lambda_i = \eta_i \circ T_{\alpha_i}$. It is clear that each $\lambda_i$ is a continuous positive functional on A. Moreover, for $x \in A$ we have

$$|\lambda_i(x)| = |\eta_i(T_{\alpha_i} x)| \leq \|(T_{\alpha_i} x)^\wedge\|_\infty \leq \|\hat{x}\|_\infty.$$ Thus $\{\lambda_i\}_{i\in I_o}$ is a net in $\Phi_1^+(A)$. Since $\Phi_1^+(A)$ is $\omega^*$-compact subset of $A'$, there exists a $\lambda \in \Phi_1^+(A)$ and a subnet $\{\lambda_i\}_{i\in I_1}$ converging to $\lambda$ in

$\omega^*$-topology.

Let us write $m = \eta \circ T$. It is clear that $m \in \Delta_\omega A$. We shall show that $\lambda(x) = m(x) \ \forall \ x \in S$.

We consider the two cases (i) $\eta = \omega$ and (ii) $\eta \in \Delta B$ separately.

Case (i) $\quad \eta = \omega$.

Let $x \in S$. Since $m(x) = \eta(Tx) = 0$ and $\lambda(x) = \lim_{i \in I_1} \lambda_i(x)$, it is enough to show that $\lim_{i \in I_1} \lambda_i(x) = 0$. Let $\epsilon > 0$.

$$\text{Now} \quad \lim_{i \in I_1} \|(T_{\alpha_i} x - Tx)^\wedge\|_\infty = 0 \text{ and } \lim_{i \in I_1} \eta_i(Tx) = \eta(Tx) = 0.$$

Therefore we can choose $i_o \in I_1$ such that $i \geq i_o$ implies $\|(T_{\alpha_i} x - Tx)^\wedge\|_\infty < \epsilon/2$ and $|\eta_i(Tx)| < \epsilon/2$. Then for $i \geq i_o$ we have

$$|\lambda_i(x)| \leq |\eta_i(T_{\alpha_i} x) - \eta_i(Tx)| + |\eta_i(Tx)| < \epsilon.$$

Case (ii) $\quad \eta \in \Delta B$.

Let $x \in S$. We must show that $\lambda(x) = m(x)$, that is, $\lim_{i \in I_1} \lambda_i(x) = m(x)$. Let $\epsilon > 0$. First, we choose $i_o \in I_1$ such that $i \geq i_o$ implies $\|(T_{\alpha_i} x - Tx)^\wedge\|_\infty < \epsilon/4$. Then we choose $i_1 \geq i_o$ such that $i \geq i_1$ implies $|\eta_i(T_{\alpha_{i_o}} x) - \eta(T_{\alpha_{i_o}} x)| < \epsilon/4$.

Now for $i \geq i_1$, we have

$$\begin{aligned} |\lambda_i(x) - m(x)| = |\eta_i(T_{\alpha_i} x) - \eta(Tx)| &\leq |\eta_i(T_{\alpha_i} x) - \eta_i(Tx)| \\ &+ |\eta_i(Tx) - \eta_i(T_{\alpha_{i_o}} x)| + |\eta_i(T_{\alpha_{i_o}} x) - \eta(T_{\alpha_{i_o}} x)| \end{aligned}$$

$$+ \left|\eta(T_{\alpha_{i_o}} x) - \eta(Tx)\right| < \epsilon.$$

Thus we have $\lambda(x) = m(x)\ \forall\ x \in S$. Since $y \in U$, it follows that $\lambda(y) = m(y)$.

Now,

$$\lim_{i\in I_1} \eta_i(Ty) = \eta(Ty) = m(y) = \lambda(y) = \lim_{i\in I_1} \lambda_i(y) = \lim_{i\in I_1} \eta_i(T_{\alpha_i} y).$$

Therefore $\lim_{i\in I_1} \left[\eta_i(T_{\alpha_i} y) - \eta_i(Ty)\right] = 0$. This contradicts (5.1.1).

Therefore we must have $\lim_{i\in I} \|(T_i y - Ty)^\wedge\|_\infty = 0$, that is, $y \in \hat{\text{Kor}}_u^{1+}(S)$. This completes the proof.

In view of Theorem 2.6.6 and Proposition 5.1.1, it is easily observed that if the Banach algebra A satisfies property '$P_1$' then Korovkin closure w.r.t. positive spectral contraction operators and Korovkin closure w.r.t. positive contraction operators are same. However, if A does not satisfy property '$P_1$' then the two notions are different, we shall see some examples in the sequel. We now establish the following simple proposition.

Proposition 5.1.2 Let A be a commutative Banach algebra with continuous symmetric involution and let S be a subset of A. Then S is a universal Korovkin set in A w.r.t. positive spectral contraction operators iff $\hat{S}$ is a universal Korovkin set in $C_o(\Delta A)$ w.r.t. positive contraction operators.

Proof Since $\hat{A}$ is dense in $C_o(\Delta A)$ '==>' part is obviously true. Conversely, suppose $\hat{S}$ is a universal Korovkin set in $C_o(\Delta A)$ w.r.t. positive contraction operators. Let $m \in \Delta_\omega A$ and $\phi \in \Phi_1^+(A)$ be such

that $\phi(x) = m(x)\ \forall\ x \in S$. In view of Proposition 5.1.1, we must show that $\phi(y) = m(y)\ \forall\ y \in A$.

Now $\phi$ is a positive functional on A satisfying $|\phi(x)| \le \|\hat{x}\|_\infty$ $\forall\ x \in A$. Therefore, there exists a positive regular Borel measure $\mu$ on $\Delta A$ such that $\mu(\hat{x}) = \phi(x)\ \forall\ x \in A$ and $\|\mu\| \le 1$. In particular, $\mu$ satisfies $\mu(\hat{x}) = \hat{x}(m)\ \forall\ \hat{x} \in \hat{S}$. Since $\hat{S}$ is a universal Korovkin set in $C_o(\Delta A)$ w.r.t. positive contraction operators, it follows that $\mu = \delta_m$. Therefore we have $\phi(y) = m(y)\ \forall\ y \in A$.
This completes the proof.

Remark 5.1.1 We remark that in Theorem 1.1(2) of [55], Romanelli has obtained similar conclusion as in Proposition 5.1.2, under the additional hypothesis that A has a bounded approximate identity. However, a slightly more general notion of positivity is considered in [55]; an element $x \in A$ is said to be positive if $\hat{x} \ge 0$ and a linear operator $T : A \longrightarrow B$ is called positive if it maps positive elements of A to positive elements of B, A and B being commutative Banach algebras. But since for a commutative Banach algebra A with continuous symmetric involution, elements of the set $\Phi_1^+(A) = \left\{\phi \in A' : \phi \text{ is positive and } |\phi(x)| \le \|\hat{x}\|_\infty \quad \forall\ x \in A\right\}$ are same with respect to both the notions of positivity, in view of Proposition 5.1.1, $\hat{\mathrm{Kor}}_u^{1+}(S)$ remain same with respect to both the definitions of positivity. Therefore, the condition of A having a bounded approximate identity is superfluous in Theorem 1.1(2) of [55].

We further remark that in Theorem 1.1(2) of [55], more general notion of T-universal Korovkin closure w.r.t. positive spectral contraction operators has been considered. But our arguments work in that general situation too and the condition of A having bounded approximate identity is again redundant.

Furthermore, we remark that the condition of A having a bounded approximate identity is also redundant in Proposition 1.10, Corollary 1.11 and Theorem 2.1 of [54].

In view of Theorem 2.6.8 and Proposition 5.1.2, we have the following corollary.

Corollary 5.1.1 Let A be a commutative Banach algebra with continuous symmetric involution. Suppose that A contains a subset S such that $\hat{S}$ separates points of $\Delta A$ strongly. Then $S \cup SS^*$ is a universal Korovkin set in A w.r.t. positive spectral contraction operators.

In view of Remark 3.1.2, the following corollary immediately follows from Corollary 5.1.1.

Corollary 5.1.2 Let A be a commutative Banach algebra with continuous symmetric involution. Then A admits a finite universal Korovkin set w.r.t. positive spectral contraction operators iff finitely many functions in $\hat{A}$ separate points of $\Delta A$.

Now we give some applications of the results developed in this chapter. Let G be locally compact abelian group and S(G) be a Segal algebra over G which is closed under the involution ~. The following question arises naturally.

Question 5.1.1 When does $S(G)$ admit a finite universal Korovkin set w.r.t. positive spectral contraction operators?

We provide a complete answer to the Question in following corollary.

Corollary 5.1.3 $S(G)$ admits a finite universal Korovkin set w.r.t. positive spectral contraction operators iff $\hat{G}$ is finite dimensional separable metric space.

Proof In view of Corollary 5.1.2 and Remark 3.1.2, necessity part is obvious. Conversely suppose $\hat{G}$ is finite dimensional separable metric space. In view of Corollary 5.1.2, it suffices to show that there exist finitely many functions in $S(G)$ such that their Fourier transforms separate points of $\hat{G}$. By Theorem 2 of Pannenberg [45], there are finitely many functions $f_1,\ldots,f_n$ in $L^1(G)$ such that $\hat{f}_1,\ldots,\hat{f}_n$ separate points of $\hat{G}$. Let $f \in S(G)$ be such that $\hat{f} > 0$. (Note that such a function exists in $S(G)$ because the maximal ideal space $\hat{G}$ is second countable and $S(G)$ has symmetric involution, see proof of Theorem 3.1.1). Now since $S(G)$ is an ideal in $L^1(G)$, the functions $f * f_1,\ldots, f * f_n$ belong to $S(G)$. It is easy to check that $\hat{f}, (f * f_1)^\wedge,\ldots,(f * f_n)^\wedge$ separate points of $\hat{G}$. In fact, if $\gamma$, $\eta$ in $\hat{G}$ satisfy $\hat{f}(\gamma) = \hat{f}(\eta)$ and $\hat{f}(\gamma)\ \hat{f}_i(\gamma) = \hat{f}(\eta)\ \hat{f}_i(\eta)\ \forall\ i = 1,\ldots,n$, then $\hat{f}_i(\gamma) = \hat{f}_i(\eta)\ \forall\ i = 1,\ldots,n$. But since $\hat{f}_1,\ldots,\hat{f}_n$ already separate points of $\hat{G}$, it follows that $\gamma = \eta$. This completes the proof.

Theorem 3.3.2 and arguments similar to those in the proof of Corollary 5.1.3 yield the following.

Corollary 5.1.4 Let $G$ be a [Z] group such that $G = KZ$ for a compact open normal subgroup $K$. Let $S(G)$ be a Segal algebra over $G$ which is closed under involution. Then following are equivalent.

(i) $Z(S(G))$ admits a finite universal Korovkin set w.r.t. positive spectral contraction operators.

(ii) $G$ is separable metrizable and $G/K$ has finite torsion free rank.

(iii) $\hat{G}$ is finite dimensional separable metric space.

We conclude the chapter with some examples. First example shows that in the absence of property '$P_1$' the two concepts, universal Korovkin set w.r.t. positive spectral contraction operators and universal Korovkin set w.r.t. positive contraction operators are different (see remarks preceeding Proposition 5.1.2).

Example 5.1.1 Let $G$ be a compact abelian metrizable group. Consider the Segal algebra $L^p(G)$ $(1 < p < \infty)$. In Example 4.1.1, we have seen that $L^p(G)$ does not satisfy property '$P_1$'. Let $\left\{\gamma_n\right\}_{n=1}^{\infty}$ be an enumeration of $\hat{G}$ and $h = \sum_{n=1}^{\infty} \frac{1}{2^n} \gamma_n$. It is clear that $h \in L^p(G)$ and $\hat{h}$ separates points of $\hat{G}$ strongly. An application of Corollary 5.1.1 shows that $\left\{ h, h * \tilde{h} \right\}$ is a universal Korovkin set in $L^p(G)$ w.r.t. positive spectral contraction operators. However, the set $\left\{ h, h * \tilde{h} \right\}$ is not a universal Korovkin set in $L^p(G)$ w.r.t. positive contraction operators (see, Example 4.1.2).

Example 5.1.2 Let $G$ be a locally compact abelian group and $S(G)$ be a Segal algebra over $G$. Let $E$ be a subset of $S(G)$. Suppose $E$ is a universal Korovkin set in $L^1(G)$ w.r.t. positive contraction operators. Then it follows from Corollary 4.1.1 that $\hat{E}$ is a universal Korovkin set in $C_o(\hat{G})$ w.r.t. positive contraction operators. In view of Proposition 5.1.2, $E$ will be a universal Korovkin set in $S(G)$ w.r.t. positive spectral contraction operators. For example, consider the function $h$ defined on $\mathbb{R}$ by

$$h(x) = \begin{cases} 0 & \text{if } x < 0 \\ \exp[(-\lambda+i\mu)x] & \text{if } x \geq 0 . \end{cases}$$

Altomare [2, example 4.7] has shown that $\left\{ h, \tilde{h} \right\}$ is a universal Korovkin set in $L^1(\mathbb{R})$ w.r.t. positive contraction operators. Since the set $\left\{ h, \tilde{h} \right\}$ is contained in the algebras $C_o(\mathbb{R}) \cap L^1(\mathbb{R})$ and $L^p(\mathbb{R}) \cap L^1(\mathbb{R})$ $(1 < p < \infty)$, it is also a universal universal Korovkin set in these Segal algebras w.r.t. positive spectral contraction operators.

# BIBLIOGRAPHY


[1] **Altomare, F.** On the Korovkin approximation theory in commutative Banach algebras, Rend. Mat. (7)2 (1982), no. 4, 755-767.

[2] **Altomare, F.** On the universal convergence sets, Ann. Mat. Pura Appl. (4) 138 (1984), 223-243.

[3] **Altomare, F.** Korovkin closures in Banach algebras, Operator Theory: Advances and Applications, Vol. 17, 35-42, Birkhäuser Verlag Basel, 1986.

[4] **Altomare, F. and Boccaccio, C.** On Korovkin-type theorems in spaces of continuous complex-valued functions, Boll. Un. Mat. Ital. (6) 1-B (1982), no. 1, 75-86.

[5] **Altomare, F. and Campiti, M.** A bibliography on the Korovkin type approximation theory (1952-1987), in : Functional Analysis and Approximation Theory, Ed. P.L. Papini, Proc. Int. Conf. Bagni di Lucca 1988, Pitagora Ed. Bologna, 1989, 34-79.

[6] **Bauer, H.** Theorems of Korovkin type for adapted spaces, Ann. Inst. Fourier (Grenoble) 23 (1973), fasc. 4, 245-260.

[7] **Bauer, H.** Approximation and abstract boundaries, Amer. Math. Monthly 85 (1978), 632-647.

[8] **Bauer, H. and Donner, K.** Korovkin approximation in $C_o(X)$, Math. Ann. 236 (1978), no. 3, 225-237.

[9] **Bauer, H. and Donner, K.** Korovkin closures in $C_o(X)$, in : Aspects of Mathematics and its applications (Ed. J.A. Barroso), 145-167. Math. Library 34, North Holland, New York 1986.

[10] **Beckhoff, F.** Korovkin-Theorie in Algebren, Schriftenr. Math. Inst. Univ. Münster 2 Ser. 45 (1987).

[11] **Berens, H. and Lorentz, G.G.** Theorems of Korovkin type for positive linear operators on Banach lattices, Approximation Theory (Proc. Internat. Sympos. Univ. Texas, Austin, Tex., 1973), 1-30; Academic Press, New York, 1973.

[12] **Berens, H. and Lorentz, G.G.** Geometric theory of Korovkin sets, J. Approx. Theory 15 (1975), no. 3, 161-189.

[13] **Bloom, W.R. and Sussich, J.F.** The degree of approximation by positive linear operators on compact connected abelian groups, J. Austral. Math. Soc. (Series A) 33 (1982), 364-373.

[14] **Briem, E.** Convergence of sequences of positive operators on $L_p$-spaces, Proceedings of the Nineteenth Nordic Congress of Mathematicians, Reykjavik, 1984, 126-131; Visindafel. Isl., XLIV, Icel. Math. Soc. Reykjavik, 1985.

[15] **Brosowski, B.** The completion of partially ordered vector spaces and Korovkin's theorem, Approximation theory and functional analysis (Proc. Internat. Sympos. Approximation Theory, Univ. Estadual de Campinas, Campinas, 1977), 63-69, North Holland Math. Studies 35, North Holland, Amsterdam, 1979.

[16] **Censor, E.** Quantitative results for positive linear approximation operators, J. Approx. Theory 4 (1971), 442-450.

[17] **Donner, K.** Korovkin theorems in $L^p$-spaces, J. Functional Analysis 42 (1981) no. 1, 12-28.

[18] **Donner, K.** Extension of positive operators and Korovkin theorems, Lecture Notes in Math. 904, Springer-Verlag, Berlin, 1982.

[19] **Flösser, H.O.** Sequences of positive contractions on AM-spaces, J. Approx. Theory 31 (1981), 118-137.

[20] **Gelfand, I.M., Raikov, D.A. and Shilov, G.E.** Commutative normed rings, Chelsea Publishing Co., New York 1964.

[21] **Grosser, S. and Moskowitz, M.** On central topological groups, Trans. Amer. Math. Soc. 127 (1967), 317-340.

[22] **Grosser, S. and Moskowitz, M.** Representation theory of central topological groups, Trans. Amer. Math. Soc. 129 (1967), 361-390.

[23] **Grosser, S. and Moskowitz, M.** Harmonic analysis on central topological groups, Trans. Amer. Math. Soc. 156 (1971), 419-454.

[24] **Grosser, S. and Moskowitz, M.** Compactness conditions in topological groups, J. Reine Angew. Math. 246 (1971), 1-40.

[25] **Grosser, S., Mosak, R. and Moskowitz, M.** Duality and Harmonic analysis on central topological groups, Indag. Math. 35 (1973), 65-91.

[26] **Hewitt, E. and Ross, K.A.** Abstract Harmonic Analysis, Springer-Verlag, Berlin, Vol. I (1963), Vol. II (1970).

[27] **Johnson, B.E.** Some examples in Harmonic Analysis, Studia Math. 48 (1973), 181-188.

[28] **Katznelson, Y.** An introduction to Harmonic Analysis, John Wiley and sons Inc. New York (1968).

[29] **Kitto, W. and Wulbert, D.E.** Korovkin approximations in $L_p$-spaces, Pacific J. Math. 63 (1976), 153-167.

[30] **Korovkin, P.P.** On convergence of linear positive operators in the space of continuous functions, Dokl. Akad. Nauk SSSR (N.S.), 90 (1953), 961-964.

[31] **Labsker, L.G.** Korovkin sets in a Banach space for sets of linear functionals, Math. Notes 31 (1982) no. 1-2, 47-56.

[32] **Larsen, R.** Banach Algebras: an introduction, Marcel Dekker, Inc., New York, 1973.

[33] **Limaye, B.V. and Namboodiri, M.N.N.** Korovkin-type approximation on $C^*$-algebras, J. Approx. Theory 34 (1982), 237-246.

[34] **Liu, Teng-Sun and Rooij, Arnoud Van** Sums and intersections of normed linear spaces, Math. Nachr. 42 (1969), 29-42.

[35] **Liukkonen, J. and Mosak, R.** Harmonic analysis and centers of group algebras, Trans. Amer. Math. Soc. 195 (1974), 147-163.

[36] **Lorentz, G.G.** Korovkin theorems for contractions and positive contractions, in : Approximation Theory and Applications, (Proc. Internat. Conf. 75th Birthday G.G. Lorentz, St. John's/Newfoundland, 1984), Res. notes in Math. no. 133, Pitman Publ., Boston, 1985, 104-114.

[37] **Maltese, G.** Integral representation theorems via Banach algebras, L'Enseigment mathématique, T. XXV, fasc. 3-4 (1979), 274-284.

[38] **Mond, B. and Vasudevan, R.** On approximation by linear positive operators, J. Approx. Theory 30 (1980), 334-336.

[39] **Mosak, R.D.** Central functions in group algebras, Proc. Amer. Math. Soc. 29 (1971), 613-616.

[40] **Nagata, Jun-iti** Modern Dimension Theory, North Holland Publishing Company, Amsterdam, 1965.

[41] **Nakamoto, R. and Nakamura, N.** On theorems of Korovkin II, Proc. Jap. Acad. 41 (1965), 433-435.

[42] **Nishishiraho, T.** The degree of convergence of positive linear operators, Tôhoku Math. J. 29 (1977), 81-89.

[43] **Pannenberg, M.** Korovkin approximation in Waelbroeck algebras, Math. Ann. 274 (1986), 423-437.

[44] **Pannenberg, M.** Topics in qualitative Korovkin approximation, Conferenze del Seminario di Matematica dell'Universitā di Bari, 226, Universitā di Bari, Bari, 1988.

[45] **Pannenberg, M.** A characterization of a class of locally compact abelian groups via Korovkin theory, Math. Z. 204 (1990), 451-464.

[46] **Pannenberg, M.** When does a commutative Banach algebra possess a finite universal Korovkin system? , Atti. Sem. Mat. Fis. Univ. Modena 40 (1992) no. 1, 89-99.

[47] **Pannenberg, M. and Romanelli, S.** Korovkin approximation of operators in commutative involutive Banach algebras, Atti. Sem. Mat. Fis. Univ. Modena 40 (1992), no. 1, 101-113.

[48] **Parthasarathy, K.** Segal Algebras - Some Explorations, Ph.D. Thesis, Indian Institute of Technology, Kanpur (INDIA), 1977.

[49] **Phelps, R.R.** Lectures on Choquet's Theorem, D. Van Nostrand Comp. Inc., 1966.

[50] **Priestley, W.M.** A noncommutative Korovkin theorem, J. Approx. Theory 16 (1976), 251-260.

[51] **Reiter, H.** Classical Harmonic analysis and locally compact groups, Oxford University Press (1968).

[52] **Reiter, H.** $L^1$-algebras and Segal algebras, Lecture Notes in Math. Vol. 231, Springer-Verlag, Berlin-Heidelberg - New York (1971).

[53] **Robertson, A.G.** A Korovkin theorem for Schwartz maps on $C^*$-algebras, Math. Z. 156 (1977), 205-207.

[54] **Romanelli, S.** Determining subspaces for discrete-type linear forms on commutative Banach algebras, Rend. Circ. Mat. Palermo (2) 38 (1989) no. 3, 455-476 (1990).

[55] **Romanelli, S.** Universal Korovkin closures with respect to linear operators on commutative Banach algebras, Math. Japon. 37 (1992), no. 3, 427-443.

[56] **Saskin, Ju. A.** Korovkin systems in spaces of continuous functions, Amer. Math. Soc. Transl. Ser. 2, 54 (1966), 125-144.

[57] **Takahasi, Sin-Ei** Korovkin's theorems for $C^*$-algebras, J. Approx. Theory 27 (1979), 197-202.

[58] **Wolf, M.** On the universal Korovkin closure of subsets in vector lattices, J. Approx. Theory 22 (1978), 243-253.